Bivariate normal densities: X and Y are said to have a **bivariate normal density** (a.k.a. **joint normal density**) if their joint density function is

$$\begin{aligned} f_{X,Y}(x,y) &= (const)\exp\left[(const)x^2 + (const)xy + (const)y^2 + (const)x + (const)y + const\right] \\ &= \frac{1}{2\pi\sqrt{\det\Sigma}}\exp\left[-\frac{a}{2}x^2 - by - \frac{d}{2}y^2 + (a\mu_X + b\mu_Y)x + (b\mu_X + d\mu_Y)y + const\right] \\ &= \frac{1}{2\pi\sqrt{\det\Sigma}}\exp\left[-\frac{1}{2}\left[\begin{pmatrix} x-\mu_X & y-\mu_Y \end{pmatrix}\Sigma^{-1}\begin{pmatrix} x-\mu_X \\ y-\mu_Y \end{pmatrix}\right]\right] \end{aligned}$$

where $\Sigma^{-1} = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$ and

$$\Sigma = \begin{pmatrix} Cov(X,X) = Var(X) & Cov(X,Y) \\ Cov(X,Y) & Cov(Y,Y) = Var(Y) \end{pmatrix}.$$

The formulas above give a method of going between a bivariate density function and the means, variances and covariances of X and Y.

Linear combinations of joint normal r.v.s are normal: If X and Y are bivariate normal, then for any constants b_1, b_2, then $b_1X + b_2Y$ is normal (and its parameters can be computed using properties of means, variances and covariances).

Conditional densities and expectations: If X and Y are bivariate normal, then $Y \mid X$ is normal with parameters

$$E(Y|X) = \mu_Y + \frac{Cov(X,Y)}{Var(X)}(x-\mu_X)Var(Y|X) = Var(Y)(1-\rho^2)$$

where ρ is the correlation between X and Y (i.e $\rho = \frac{Cov(X,Y)}{\sqrt{Var(X)\cdot Var(Y)}}$). Similarly, $X \mid Y$ is normal with parameters

$$E(X|Y) = \mu_X + \frac{Cov(X,Y)}{Var(Y)}(y-\mu_Y) \quad \text{and} \quad Var(X|Y) = Var(X)(1-\rho^2)$$

If X and $Y|X$ are normal, then X and Y have a bivariate normal density.

Sample problems:

1. Suppose X and Y have a bivariate normal density where $EX = 3$, $Var(X) = 2$, $EY = -4$ and $Var(Y) = 6$. If $Cov(X, Y) = 2$,
   (a) find the joint density of X and Y;
   (b) find the density function of $2X + Y$;
   (c) find the probability that $X - Y \leq 7.5$;
   (d) find the conditional expectation of Y given X;
   (e) find the conditional variance of Y given X.
2. Suppose X is normal with mean 0 and variance 5, and suppose that $Y|X$ is normal with conditional expectation $2x + 1$ and conditional variance 8.
   (a) Find the covariance between X and Y;
   (b) find the correlation between X and Y;
   (c) find the mean and variance of Y;
   (d) find the probability that $Y \geq 0$;
   (e) find the probability that $X + 2Y \leq 3$.