1. Give precise definitions of three of the following four terms.
   (a) Cauchy sequence (in a metric space)
   (b) compact
   (c) limit superior
   (d) open set (in a metric space)

2. Precisely state three of the following four theorems.
   (a) Supremum Property
   (b) Bolzano-Weierstrass Theorem
   (c) Lindelöf's Theorem
   (d) Heine-Borel Theorem

3. Classify any five of the following six statements as true or false:
   (a) If $f : X \to Y$ is some function and $A \subseteq X$, then $f(A^C) = [f(A)]^C$.
   (b) The set of real numbers in $[0, 1]$ whose decimal expansions contain only the digits 3 and 6 is a countable set.
   (c) If $S \subseteq \mathbb{R}$ is nonempty and bounded above and if $s = \sup S$, then for every $\epsilon > 0$, $s - \epsilon \in S$.
   (d) If $\{a_n\}$ is a properly divergent sequence of real numbers, then every subsequence of $\{a_n\}$ diverges.
   (e) The union of any finite number of bounded subsets in a metric space is bounded.
   (f) A subset of a metric space is closed if and only if it contains all its cluster points.

4. Prove four of these six statements; you must prove at least one of (a) and (b), at least one of (c) and (d), and at least one of (e) and (f).
   (a) The Cantor set C is in $1 - 1$ correspondence with $\mathbb{R}$.
   (b) $\mathbb{R}$ is uncountable.
   (c) If $\{a_n\}$ is a bounded sequence of real numbers, with the property that every convergent subsequence of $\{a_n\}$ converges to L, then $a_n \to L$.
   (d) If $\{a_n\}$ and $\{b_n\}$ are two convergent sequences of real numbers, then $\{a_n + b_n\}$ converges.
   (e) Every complete subset of a metric space is closed.
   (f) Every compact metric space is complete.

5. Prove each of the following three statements.
   (a) If A and B are nonempty subsets of $\mathbb{R}$, each bounded above, then $A \cup B$ is bounded above and $\sup(A \cup B) = \sup\{\sup A, \sup B\}$.
   (b) If $\{x_n\}$ is the sequence of real numbers defined recursively by setting $x_1 = 1$ and $x_n = \sqrt{2 + x_{n-1}}$ for all $n \geq 2$, then $\{x_n\}$ converges.
   (c) A sequentially compact metric space must be totally bounded.

6. Prove any one of the following three statements:
   (a) $\mathbb{R}$ is in $1 - 1$ correspondence with $\mathbb{R}^2$.
   (b) If $\{x_n\}$ is a bounded sequence of real numbers, then the set S of subsequential limits of $\{x_n\}$ is a closed subset of $(\mathbb{R}, |\cdot|)$.
   (c) There exists a metric space which is complete and bounded, but not compact.

1. (a) Let (X, d) be a metric space. A sequence $\{x_n\}$ of points in X is called a *Cauchy sequence* if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that whenever $m, n \geq N$, $d(x_m, x_n) < \epsilon$.

   (b) Let X be a topological space. A set $A \subseteq X$ is called *compact* if given any collection $\{U_\alpha : \alpha \in I\}$ of open sets with $A \subseteq \bigcup_{\alpha \in I} U_\alpha$, then there exists a finite set $\{\alpha_1, ..., \alpha_n\} \subseteq I$ such that $A \subseteq \bigcup_{j=1}^n U_{\alpha_j}$.

   (c) Let $\{a_n\}$ be a bounded sequence of real numbers. Define the *limit superior* of the sequence to be the real number

$$\overline{\lim}\, a_n = \sup\{y : y < a_n \text{ for infinitely many } n\}.$$

   (d) Let d be a metric on X. We say a set $U \subset X$ is *open* (with respect to d) if for every $x \in U$, there exists $\epsilon > 0$ such that $B_\epsilon(x) \subseteq U$.

2. (a) **Supremum Property:** Let $S \subseteq \mathbb{R}$ be nonempty and bounded above. Then $\sup S$ exists.

   (b) **Bolzano-Weierstrass Theorem:** Every bounded sequence of real numbers has a convergent subsequence.

   (c) **Lindelöf's Theorem:** A set $U \subseteq \mathbb{R}$ is open (with respect to the usual metric) if and only if it is a countable union of disjoint open intervals.

   (d) **Heine-Borel Theorem:** A subset of $\mathbb{R}^n$ (with the Euclidean metric) is compact if and only if it is closed and bounded.

3. (a) FALSE; suppose f is a constant function and $A = X$. Then $f(A^C) = f(\emptyset) = \emptyset$ but $[f(A)]^C \neq \emptyset$ unless Y has only one element.

   (b) FALSE; this set is in $1-1$ correspondence with the Cantor set (send the number with decimal expansion $.x_1x_2x_3...$ to the number with ternary expansion $.y_1y_2y_3....$ where $y_j = 0$ if $x_j = 3$ and $y_j = 2$ if $x_j = 6$; this gives a bijection).

   (c) FALSE; set $S = \{0\}$, then $\sup S = 0$, but for every $\epsilon > 0$, $(-\epsilon, 0) \cap S = \emptyset$.

   (d) TRUE; if a subsequence converges, then that subsequence is bounded, but by defintion of "properly divergent sequence", no subsequence of $\{a_n\}$ can be bounded (for every M, there exists $N \in \mathbb{N}$ such that $|a_n| > M$ for all $n \geq N$).

   (e) TRUE; every bounded subset is contained in an open ball, and it was shown in the homework that a finite union of open balls is bounded.

   (f) TRUE; this was a homework problem.

4. (a) Let C denote the Cantor set. Since $C \subseteq [0, 1]$, the identity map gives an injection from C into $[0, 1]$. Let G be the set of real numbers in $[0, 1]$ which have a unique binary representation. This set is infinite as it contains all irrational numbers, and since $[0, 1] - G$ is countable (HW problem), we see $[0, 1] \sim G$, i.e. there is a bijection $f : [0, 1] \to G$. Now, define $h : G \to [0, 1]$ by setting $h(x)$ to be the real number in $[0, 1]$ with ternary representation $.(2x_1)(2x_2)(2x_3)...$, where x has binary representationn $.x_1x_2x_3.....$ Since the digits in the ternary representation of $h(x)$ are all 0 or 2, we see $h(G) \subseteq C$. Furthermore, h is $1-1$: if $h(x) = h(y)$ where $x \neq y$, then the ternary representations $.(2x_1)(2x_2)(2x_3)...$ and $.(2y_1)(2y_2)(2y_3)...$

must represent the same number. But two different ternary representations can only represent the same number if one of them contains a 1.

At this point, we have $1-1$ maps $C \to [0,1]$ (the identity) and $[0,1] \to C$ (the function $h \circ f$), so by the Cantor-Bernstein Theorem, $C \sim [0,1]$. By homework problem, $[0,1] \sim \mathbb{R}$ so we are done.

(b) Since $[0,1] \sim \mathbb{R}$, it is sufficient to show $[0,1]$ is uncountable. Suppose not, then enumerate the elements of $[0,1]$ as $\{z_1, z_2, z_3, ...\}$ and take any decimal representation of each of the z_j; we have $z_j = .z_{j1}z_{j2}z_{j3}...$ for all j. Now define, for each j,

$$y_j = \begin{cases} 2 & \text{if } z_{jj} \geq 5 \\ 7 & \text{if } z_{jj} < 5 \end{cases}$$

and let $y \in [0,1]$ be the real number with decimal representation $.y_1y_2y_3....$. Since this decimal representation does not end with an infinite string of 0s or 9s, it is the only decimal representation of y.

Notice, however, that $y_j \neq z_{jj}$ for all j, and therefore $y \neq z_j$ for all j. Therefore we did not in fact enumerate all the elements of $[0,1]$, so $[0,1]$ (and also $\mathbb{R}$) is uncountable.

(c) Suppose not, i.e. $\{a_n\}$ does not converge to L. Then by a result from class, there exists $\epsilon_0 > 0$ and a subsequence $\{a_{n_k}\}$ such that $d(a_{n_k}, L) \geq \epsilon_0$ for all k. This subsequence is bounded (since $\{a_n\}$ was), so by the Bolzano-Weierstrass Theorem, there exists a subsequence of $\{a_{n_k}\}$ (call it $\{a_{n_{k_l}}\}$) which converges to some L'. Since $d(a_{n_{k_l}}, L) \geq \epsilon_0$ for all k, $d(L', L) \geq \epsilon_0$, i.e. $L \neq L'$, contradicting the hypothesis.

(d) Let $L = \lim a_n$ and $M = \lim b_n$. We claim $(a_n + b_n) \to L + M$. To show this, first fix $\epsilon > 0$. Choose $N_a \in \mathbb{N}$ such that if $n \geq N_a$, then $d(a_n, L) = |a_n - L| < \epsilon/2$. Choose $N_b \in \mathbb{N}$ such that if $n \geq N_b$, then $d(b_n, M) = |b_n - M| < \epsilon/2$. Let $N = \max(N_a, N_b)$. For $n \geq N$, we have

$$d(a_n + b_n, L + M) = |a_n + b_n - L - M| \leq |a_n - L| + |b_n - M| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

so $(a_n + b_n) \to L + M$ by definition of convergence.

(e) Let (X, d) denote the metric space and let $A \subseteq X$ be complete. Let $\{x_n\}$ be a sequence of points in A which converges to $x \in X$. Since $\{x_n\}$ converges, it is Cauchy and since A is complete, $\{x_n\}$ converges to a point in A. By uniqueness of limits, this means $x \in A$. Thus A is "sequentially closed", which is equivalent to A being closed by a homework problem.

(f) Let (X, d) denote the compact metric space. Let $\{x_n\}$ be a Cauchy sequence of points in X; since (X, d) is compact, it is sequentially compact so there exists a convergent subsequence $\{x_{n_k}\}$. But if a Cauchy sequence has a convergent subsequence then it must converge itself, so $\{x_n\}$ converges. Therefore (X, d) is complete by definition (every Cauchy sequence converges).

5. (a) Suppose A is bounded above by r and B is bounded above by s. Then $A \cup B$ is bounded above by $\sup(r, s)$, for if $x \in A \cup B$, then either $x \in A$ (in which case $x \leq r \leq \sup(r, s)$) or $x \in B$ (in which case $x \leq s \leq \sup(r, s)$).

The preceding paragraph shows that $q = \sup\{\sup A, \sup B\}$ is an upper bound for $A \cup B$. Suppose p is any upper bound for $A \cup B$. Then p is also an upper bound for A, so $p \geq \sup A$. Also, p must be an upper bound for B, so $p \geq \sup B$. Therefore p is an upper bound for $\{\sup A, \sup B\}$ so $p \geq q$. By definition, $q = \sup(A \cup B)$.

(b) We first claim that $\{x_n\}$ is bounded above by 2. The proof is by induction. Clearly $x_1 = 1 \leq 2$; now suppose $x_k \leq 2$. Then $x_{k+1} = \sqrt{2 + x_k} \leq \sqrt{2+2} = 2$. By induction, we conclude $x_n \leq 2$ for all n.

Next, we claim that $x_n > 0$ for all n. This is clear, by induction ($x_1 = 1 > 0$, and so long as x_k is positive, so is x_{k+1}). Therefore we now know $x_n \in (0, 2]$ for all n.

We next claim $\{x_n\}$ is increasing (there are many different ways to prove this). First, since $x_n > 0$, $x_n(x_n - 1)$ has the same sign as $x_n - 1$. If $x_n - 1 \geq 0$, then $x_n(x_n - 1) \leq 2 \cdot 1 = 2$, and if $x_n - 1 < 0$, then $x_n(x_n - 1) < 0 < 2$. In either case, we have shown that $x_n(x_n - 1) \leq 2$. Multiply out the left-hand side to obtain $x_n^2 - x_n \leq 2$, i.e. $x_n^2 \leq 2 + x_n$; take square roots to obtain $x_n \leq \sqrt{2 + x_n} = x_{n+1}$. Therefore $\{x_n\}$ is increasing.

Finally, since $\{x_n\}$ is increasing and bounded above, we see that $\{x_n\}$ converges by the Monotone Convergence Theorem.

(c) Let X be a sequentially compact metric space. If $X = \emptyset$, then X is trivially totally bounded (it is covered by zero balls of any fixed radius). So henceforth we assume X is nonempty.

Suppose X is not totally bounded. Then there exists $\epsilon > 0$ such that X cannot be covered by a finite number of balls of radius ϵ. Let $x_1 \in X$. Since $B_\epsilon(x_1) \neq X$, there exists $x_2 \in X - B_\epsilon(x_1)$. Now for each $n \geq 3$, we can choose $x_n \in X - \bigcup_{j=1}^{n-1} B_\epsilon(x_j)$ (since $\bigcup_{j=1}^{n-1} B_\epsilon(x_j)$ cannot be all of X). Thus we obtain a sequence $\{x_n\}$ satisfying $d(x_k, x_l) \geq \epsilon$ for all k, l. This sequence cannot have a convergent subsequence because no subsequence can be Cauchy (there is no $m, n \in \mathbb{N}$ such that $d(x_m, x_n) < \frac{\epsilon}{2}$). This contradicts the hypothesis that X is sequentially compact. Hence X must be totally bounded.

6. (a) We claim $2^{\mathbb{N}} \sim (2^{\mathbb{N}} \times 2^{\mathbb{N}})$. To prove this claim, first establish some notation: for a real number r and a set $A \subseteq \mathbb{N}$, set $rA = \{ra \in \mathbb{N} : a \in A\}$. For example, $2\mathbb{N}$ is the set of even natural numbers. Next, for an integer q and a set $A \subseteq \mathbb{N}$, set $A + q = \{a + q \in \mathbb{N} : a \in A\}$. For example, $2\mathbb{N} - 1$ is the set of odd natural numbers.

Now, define a function $g : (2^{\mathbb{N}} \times 2^{\mathbb{N}}) \to 2^{\mathbb{N}}$ by setting $g(A, B) = 2A \cup (2B - 1)$. Also define $h : 2^{\mathbb{N}} \to (2^{\mathbb{N}} \times 2^{\mathbb{N}})$ by

$$h(C) = \left(\frac{1}{2}(C \cap 2\mathbb{N}), \frac{1}{2}[(C + 1) \cap 2\mathbb{N}] \right).$$

We claim g and h are inverses. First, let $C \in 2^{\mathbb{N}}$. Then

$$\begin{aligned} g(h(C)) &= g\left(\frac{1}{2}(C \cap 2\mathbb{N}), \frac{1}{2}[(C+1) \cap 2\mathbb{N}]\right) \\ &= 2\left[\frac{1}{2}(C \cap 2\mathbb{N})\right] \bigcup 2\left[\frac{1}{2}[(C+1) \cap 2\mathbb{N}]\right] \\ &= (C \cap 2\mathbb{N}) \cup [(C+1) \cap 2\mathbb{N}] \\ &= (C \cap 2\mathbb{N}) \cup (C \cap (2\mathbb{N}-1)) \\ &= C \cap (2\mathbb{N} \cup (2\mathbb{N}-1)) = C \cap \mathbb{N} = C. \end{aligned}$$

Now let $A, B \in 2^{\mathbb{N}}$. Then

$$\begin{aligned} h(g(A,B)) &= h(2A \cup (2B-1)) \\ &= \left(\frac{1}{2}([2A \cup (2B-1)] \cap 2\mathbb{N}), \frac{1}{2}[([2A \cup (2B-1)] + 1) \cap 2\mathbb{N}]\right) \\ &= \left(\frac{1}{2}(2A \cap 2\mathbb{N}), \frac{1}{2}[2B \cap 2\mathbb{N}]\right) \\ &= (A \cap \mathbb{N}, B \cap \mathbb{N}) = (A, B). \end{aligned}$$

Therefore g and h are inverses of one another; hence g is a bijection so $2^{\mathbb{N}} \sim (2^{\mathbb{N}} \times 2^{\mathbb{N}})$. But we know from the homework that $\mathbb{R} \sim 2^{\mathbb{N}}$, so $(\mathbb{R} \times \mathbb{R}) \sim (2^{\mathbb{N}} \times 2^{\mathbb{N}})$ by taking a bijection on each coordinate. Therefore we conclude

$$\mathbb{R} \sim 2^{\mathbb{N}} \sim (2^{\mathbb{N}} \times 2^{\mathbb{N}}) \sim (\mathbb{R} \times \mathbb{R}) = \mathbb{R}^2.$$

(b) We will prove S is closed by showing S^C is open. Let $y \in S^C$; thus no subsequence of $\{x_n\}$ converges to y by definition.

Lemma: $y \in S^C \Rightarrow$ there exists $\epsilon > 0$ such that $B_\epsilon(y) \cap \{x_n\} = \emptyset$.

Proof of Lemma: Suppose not, i.e. $B_\epsilon(y) \cap \{x_n\} \neq \emptyset$ for every $\epsilon > 0$.

We will first show that for every $\epsilon > 0$, there are infinitely many n such that $x_n \in B_\epsilon(y)$. For if there are only finitely many n, say $n_1, ..., n_r$, such that $x_{n_k} \in B_\epsilon(y)$, choose $\delta = \frac{1}{2}\min\{d(x_{n_k}, y) : k \in \{1, ..., r\}\}$; then $B_\delta(y) \cap \{x_n\}$ is empty, contradicting the hypothesis. Thus for every $\epsilon > 0$, $B_\epsilon(y) \cap \{x_n\}$ is infinite.

Next, define a subsequence $\{x_{n_k}\}$ recursively as follows: first, there exists $x_{n_1} \in B_1(y)$. Then, for each $k \geq 1$, set $\delta_k = \frac{1}{2}d(x_{n_k}, y)$; by the previous paragraph there exists $n_{k+1} > n_k$ such that $x_{n_{k+1}} \in B_{\delta_k}(y)$. For every $k \geq 1$, we have

$$d(x_{n_k}, y) < \delta_k < \frac{1}{2}d(x_{n_{k-1}}, y)$$

so by repeating this argument we see

$$d(x_{n_k}, y) < \frac{1}{2}d(x_{n_{k-1}}, y) < \frac{1}{4}d(x_{n_{k-2}}, y) < \cdots < \frac{1}{2^{k-1}}d(x_{n_1}, y) = \frac{1}{2^{k-1}}.$$

Therefore by the Squeeze Theorem, $d(x_{n_k}, y) \to 0$ as $k \to \infty$, hence $x_{n_k} \to y$ and $y \in S$, contradicting the hypothesis of the lemma.

Having proven the lemma, let $\epsilon > 0$ be such that the open set $U = B_\epsilon(y)$ contains no points of $\{x_n\}$. Thus the closed set U^C contains all points in the sequence, thus contains all subsequences of $\{x_n\}$, and since U^C is closed it contains all limits of subsequences of $\{x_n\}$ (i.e. $S \subseteq U^C$). Thus $U = B_\epsilon(y) \subseteq S^C$ so S^C is open so S is closed.

(c) *First example:* Let d be the discrete metric on any infinite set X, i.e. set $d(x, y) = 1$ if $x \neq y$ and $d(x, y) = 0$ if $x = y$. This is a metric, and X is bounded (as $X \subseteq B_2(x)$ for any $x \in X$). Since X is infinite, (X, d) is not compact by a homework problem. Last, we show (X, d) is complete by verifying that every Cauchy sequence converges. Suppose $\{x_n\} \subseteq X$ is Cauchy. Then, for $\epsilon = \frac{1}{2}$, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$, $d(x_m, x_n) < \frac{1}{2}$. But since d is discrete, this means that for all $n \geq N$, $x_n = x_N$, i.e. the sequence is constant beyond the N^{th} term. Therefore $x_n \to x_N$.

Second example: First, let d denote the usual metric on $\mathbb{R}$. Now, let d^* be the nearsighted metric d^* defined by $d^*(x, y) = \min(1, d(x, y))$ for $x, y \in \mathbb{R}$). Observe that d and d^* are equivalent metrics, because any open d^*-ball of radius r contains the open $d-$ball of radius r (centered at the same point), and any open $d-$ball of radius r contains the open d^*-ball of radius $\min(r, \frac{1}{2})$ (centered at the same point). That d^* is a metric was prove in the homework; the metric space $(\mathbb{R}, d^*)$ will be an example which proves the result.

First, we show $(\mathbb{R}, d^*)$ is not compact. We know $(\mathbb{R}, d)$ is not compact, hence there exists an cover $\{U_\alpha\}$ of $\mathbb{R}$ by $d-$open sets which has no finite subcover. But since d^* and d are equivalent metrics, we know that d^*-open sets are exactly the same as $d-$open sets, so $\{U_\alpha\}$ is also a cover of $\mathbb{R}$ by d^*-open sets lacking a finite subcover.

Second, $(\mathbb{R}, d^*)$ is bounded: since $d^*(x, y) \leq 1$ for all $x, y \in \mathbb{R}$, we see $\mathbb{R} \subseteq B_2(0)$, the d^*-open ball of radius 2 centered at 0.

Last, we show $(\mathbb{R}, d^*)$ is complete. Suppose $\{x_n\}$ is a d^*-Cauchy sequence. We claim $\{x_n\}$ is also $d-$Cauchy. For given $\epsilon > 0$, choose $N \in \mathbb{N}$ such that whenever $m, n \geq N$, then $d^*(x_m, x_n) < \min(\frac{1}{2}, \epsilon)$. Then for $m, n \geq N$, we have $d(x_m, x_n) = d^*(x_m, x_n) < \epsilon$ as desired. Now since $(\mathbb{R}, d)$ is complete, there exists $x \in \mathbb{R}$ such that $x_n \to x$ with respect to d, i.e. for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $n \geq N \Rightarrow d(x_n, x) < \epsilon$. But we see that for all n greater than or equal to the same N, $d^*(x_n, x) \leq d(x_n, x) < \epsilon$, so $x_n \to x$ with respect to d^* as well. Thus $(\mathbb{R}, d^*)$ is complete, as desired.