1. Precisely define four of the following five terms:
   (a) Riemann integrable / Riemann integral
   (b) countable set
   (c) uniformly continuous
   (d) Cauchy sequence
   (e) Taylor series
2. Precisely state any four of the following five theorems:
   (a) Monotone Convergence Theorem
   (b) Weierstrass Approximation Theorem
   (c) Intermediate Value Theorem
   (d) Mean Value Theorem
   (e) Weierstrass $M-$Test
3. Classify any five of the following six statements as true or false:
   (a) Every infinite set contains a countably infinite subset.
   (b) If $f : X \to Y$ is uniformly continuous, then for any subset $Z \subseteq X$, f is uniformly continuous on Z.
   (c) If $f : X \to Y$ is continuous, then for any Cauchy sequence $\{x_n\} \subseteq X$, $\{f(x_n)\}$ is a Cauchy sequence in Y.
   (d) Let $f \in R([a, b])$. If $\mathcal{P}$ is a partition of $[a, b]$ such that $U(f; \mathcal{P}) - \int_a^b f < \epsilon$, then for every partition $\mathcal{Q}$ of $[a, b]$ with $||\mathcal{Q}|| < ||\mathcal{P}||$, $U(f; \mathcal{Q}) - \int_a^b f < \epsilon$.
   (e) There is a differentiable function $f : [0, 1] \to \mathbb{R}$ such that f' is equal to the Cantor function (a.k.a. devil's staircase) on $(0, 1)$.
   (f) If $\{f_n\}$ is a sequence of continuous functions from $\mathbb{R}$ to $\mathbb{R}$ such that $\sum f_n$ converges to f, then f is continuous.
   (g) If a power series $\sum a_n x^n$ diverges when $x = 3$, then that power series diverges for all $x > 3$.
4. Each of the following five statements is **false**. Your task is to provide, for any three of the five statements, a **specific** counterexample which demonstrates that statement to be false.
   (a) Let (X, d) be a metric space. If B_1 and B_2 are open balls in this metric space with $B_1 \subseteq B_2$, then the radius of B_1 is less than or equal to the radius of B_2.
   (b) If $f : [a, b] \to R$ is such that $f^2 \in R([a, b])$, then $f \in R([a, b])$.
   (c) If $\{f_n\}$ is a sequence of bounded functions $[0, 1] \to \mathbb{R}$ with $f_n \to f$, then f is also bounded.
   (d) If $f : \mathbb{R} \to \mathbb{R}$ is a function, then there exists an open interval $I \subseteq \mathbb{R}$ such that $f|_I$ is bounded.
   (e) If $\{a_n\}$ is a sequence of real numbers such that $\lim_{n\to\infty}(a_{n+1} - a_n) = 0$, then $\{a_n\}$ converges.
5. Prove any three of the following six statements:
   (a) The **Preservation of Compactness**, which says that if X and Y are topological spaces and $f : X \to Y$ is continuous, then for every compact subset $K \subseteq X$, $f(K)$ is compact in Y.

(b) The **Product Rule**, which says that if $f, g : \mathbb{R} \to \mathbb{R}$ are two differentiable functions, then fg is differentiable and $(fg)' = f' \cdot g + g' \cdot f$.

(c) If $U \subseteq \mathbb{R}$ is an open interval and $f : U \to \mathbb{R}$ is differentiable on U with $f'(x) > 0$ for all $x \in U$, then f is strictly increasing on U.

(d) If $f \in R([a,b])$, then for any partition $\mathcal{P}$ of $[a,b]$, $L(f;\mathcal{P}) \leq \int_a^b f$.

(e) Let $a < b$ and suppose $f, g : [a,b] \to \mathbb{R}$ are continuous functions such that $\int_a^b f = \int_a^b g$. Prove that there exists $c \in (a,b)$ such that $f(c) = g(c)$.

(f) The **Comparison Test for Series**, which says that if $0 \leq a_n \leq b_n$ for all n, then if $\sum b_n$ converges, so does $\sum a_n$.

6. Prove any two of the following four statements:

(a) For all real numbers x and y, $|\cos x - \cos y| \leq |x - y|$.

(b) If $f : \mathbb{R} \to \mathbb{R}$ is differentiable at x_0, then for all nonzero $\alpha, \beta \in \mathbb{R}$,

$$\lim_{h \to 0} \frac{f(x_0 + \alpha h) - f(x_0 + \beta h)}{h}$$

exists.

(c) Let $f, g : \mathbb{R} \to \mathbb{R}$; let $a < b$ and let $z \in [a,b]$. Suppose $f \in R([a,b])$ and that $g(x) = f(x)$ for all $x \in [a,b] - \{z\}$. Prove $g \in R([a,b])$ and $\int_a^b g = \int_a^b f$.

(d) If $f : [0,1] \to \mathbb{R}$ is continuous and $\int_0^x f(t)\,dt = \int_x^1 f(t)\,dt$ for all $x \in [0,1]$, then $f(x) = 0$ for all $x \in [0,1]$.

7. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x^2}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

(a) Prove f is continuous.

(b) Prove f is differentiable at 0, and calculate $f'(0)$.
Remark: By the Product Rule, for all $x \neq 0$ the derivative of f exists and is

$$f'(x) = 2x \sin\left(\frac{1}{x^2}\right) - \frac{2}{x} \cos\left(\frac{1}{x^2}\right).$$

(c) True or false (with proof): $\int_0^1 f'(x)\,dx = f(1) - f(0)$.

8. Solve any one of the following four problems (for bonus points, solve more than one):

(a) Prove that if $\{x_n\}, \{y_n\} \subseteq X$ are two Cauchy sequences in metric space (X,d), then $\{d(x_n, y_n)\}$ converges in $\mathbb{R}$.

(b) Prove that if $f : \mathbb{R} \to \mathbb{R}$ is a twice-differentiable function with $f''(x) = f(x)$ and $|f(0)| = |f'(0)|$, then $|f(x)| = |f'(x)|$ for all $x \in \mathbb{R}$.

(c) Let $\alpha > 0$ be a constant. Compute, with justification, the following limit:

$$\lim_{n \to \infty} \frac{1^\alpha + 2^\alpha + 3^\alpha + \ldots + n^\alpha}{n^{\alpha+1}}.$$

(d) Prove that if $\sum_{n=1}^\infty a_n$ is an absolutely convergent series, then

$$\left| \sum_{n=1}^\infty a_n \right| \leq \sum_{n=1}^\infty |a_n|.$$

1. (a) Let $a < b$ and let $f : [a,b] \to \mathbb{R}$ be a function. We say f is *Riemann integrable* on $[a,b]$ if there exists a number $\int_a^b f$, called the *Riemann integral of f from a to b*, such that for every $\epsilon > 0$, there is a $\delta > 0$ such that if $\mathcal{P}$ is any tagged partition with $||\mathcal{P}|| < \delta$, then $\left| RS(f;\mathcal{P}) - \int_a^b f \right| < \epsilon$.

   (b) A set is *countable* if there exists a bijection between that set and some subset of $\mathbb{N}$.

   (c) Let X and Y be metric spaces. A function $f : X \to Y$ is *uniformly continuous* if for every $\epsilon > 0$, there is $\delta > 0$ such that $d(x,y) < \delta \Rightarrow d(f(x), f(y)) < \epsilon$.

   (d) Let X be a metric space. A sequence $\{x_n\} \subseteq X$ is called *Cauchy* if for every $\epsilon > 0$, there is $N \in \mathbb{N}$ such that $m,n \geq N \Rightarrow d(x_m, x_n) < \epsilon$.

   (e) Let $U \subseteq \mathbb{R}$ be open and let $f : U \to \mathbb{R}$ be an infinitely differentiable at $a \in U$. The *Taylor series of f centered at a* is the power series

$$\sum_{j=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + ...$$

2. (a) If $\{a_n\}$ is a sequence of real numbers which is monotone and bounded, then $\{a_n\}$ converges.

   (b) Given any continuous function $f : \mathbb{R} \to \mathbb{R}$ and any $\epsilon > 0$, there is a polynomial $p \in \mathbb{R}[x]$ such that $||f,p|| < \epsilon$.

   (c) Let $f : [a,b] \to \mathbb{R}$ be continuous. Then for any γ between $f(a)$ and $f(b)$, there is a $c \in (a,b)$ such that $f(c) = \gamma$.

   (d) Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Then there is $c \in (a,b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

   (e) Let $\{f_n\} \subseteq \mathbb{R}^X$ be a sequence of real-valued functions and let $\{M_n\}$ be a sequence of positive numbers such that for all n, $|f_n(x)| \leq M_n$ for all $x \in X$ and $\sum M_n$ converges. Then $\sum f_n$ converges uniformly on X.

3. (a) TRUE; this was Problem 8 of Homework 1.

   (b) TRUE; follows from definition of uniform continuity.

   (c) FALSE; let $f : (0,\infty) \to (0,\infty)$ be the continuous function $f(x) = \frac{1}{x}$; let $x_n = \frac{1}{n}$. $\{x_n\}$ is Cauchy but $\{f(x_n)\}$ is not.

   (d) FALSE; let $f : [0,1] \to \mathbb{R}$ be equal to 1 when $x \leq \frac{1}{2}$ and equal to 0 when $x > \frac{1}{2}$. Let $\mathcal{P} = \{0, \frac{1}{2}, 1\}$, then $||\mathcal{P}|| = \frac{1}{2}$ and $U(f;\mathcal{P}) = \frac{1}{2} = \int_0^1 f$. But for any partition of $[0,1]$ not containing $\frac{1}{2}$, the upper sum relative to that partition is strictly greater than $\frac{1}{2}$, hence the difference between this upper sum and the value of the integral cannot be less than ϵ if ϵ is sufficiently small.

   (e) TRUE; the Cantor function is continuous (Homework 14) and every continuous function has an antiderivative by the Fundamental Theorem of Calculus.

   (f) FALSE; let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by setting $f_n(x) = \frac{x^2}{(1+x^2)^n}$. For each x, $\sum f_n(x)$ is a geometric series so by summing that series one can see $\sum f_n = f$ where $f(x) = 0$ if $x = 0$ and $f(x) = 1 + x^2$ if $x \neq 0$. f is not continuous at 0.

   (g) TRUE; notice that this power series is centered at 0. Since it diverges at $x = 3$, its radius of convergence can be at most 3, so it must diverge whenever $|x - 0| > 3$, including all $x > 3$.

4. (a) Let $X = [0,1]$ with the usual metric. Let $B_1 = B_{1/2}(1) = (\frac{1}{2}, 1]$ and let $B_2 = B_{3/8}(3/4) = (\frac{3}{8}, 1]$. Observe $B_1 \subseteq B_2$ despite having a larger radius.

   (b) Let $f : [0,1] \to \mathbb{R}$ be defined by setting $f(x) = 1$ if $x \in \mathbb{Q}$ and $f(x) = -1$ if x is irrational. f is not Riemann integrable since it is nowhere continuous (Lebesgue criterion), but f^2 is the constant function 1 which is Riemann integrable.

   (c) Define the sequence $\{f_n\}$ by

$$f_n(x) = \begin{cases} \frac{1}{x} & \text{if } x > \frac{1}{n} \\ n & \text{if } 0 < x \leq \frac{1}{n} \\ 0 & \text{if } x = 0 \end{cases}$$

   Each $\{f_n\}$ is bounded by n, but $f_n \to f$ where $f[0,1] \to \mathbb{R}$ is the unbounded function

$$f(x) = \begin{cases} \frac{1}{x} & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

   (d) Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} n & \text{if } x = \frac{m}{n} \in \mathbb{Q} \text{ where } \gcd(m,n) = 1 \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}.$$

   Every open interval contains rational numbers of arbitrarily large denominator (reason is similar to the logic in Problem 7 of Homework 11), so f is not bounded on any open interval.

   (e) Let a_n be the n^{th} partial sum of the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$. $\{a_n\}$ diverges but $a_{n+1} - a_n = \frac{1}{n}$ which converges to zero.

5. (a) Let $\{U_\alpha\}$ be an open cover of $f(K)$. Then $\{f^{-1}(U_\alpha)\}$ is an open cover of $f^{-1}(f(K)) \supseteq K$. By compactness of K, there is a finite subcover $\{f^{-1}(U_j)\}_{j=1}^{n}$ of $\{f^{-1}(U_\alpha)\}$ which covers K, i.e. $K \subseteq \bigcup_{j=1}^{n} f^{-1}(U_j)$. Then $f(K) \subseteq f\left(\bigcup_{j=1}^{n} f^{-1}(U_j)\right) = \bigcup_{j=1}^{n}(f \circ f^{-1})(U_j) = \bigcup_{j=1}^{n} U_j$. Thus we have found a finite subcover of $\{U_\alpha\}$ which covers $f(K)$, so $f(K)$ is compact as desired.

   (b) This follows from a direct calculation of the derivative by the definition. Let $x_0 \in \mathbb{R}$:

$$\begin{aligned}(fg)'(x_0) &= \lim_{x \to x_0} \frac{(fg)(x) - (fg)(x_0)}{x - x_0} \\ &= \lim_{x \to x_0} \frac{f(x)g(x) - f(x_0)g(x_0)}{x - x_0} \\ &= \lim_{x \to x_0} \frac{f(x)g(x) - f(x_0)g(x) + f(x_0)g(x) - f(x_0)g(x_0)}{x - x_0} \\ &= \lim_{x \to x_0} g(x)\frac{f(x) - f(x_0)}{x - x_0} + \lim_{x \to x_0} f(x_0)\frac{g(x) - g(x_0)}{x - x_0} \\ &= g(x_0)f'(x_0) + f(x_0)g'(x_0).\end{aligned}$$

   The last line follows from the assumption that f and g are differentiable at x_0.

   (c) Suppose not, i.e. there is $a, b \in U$ with $a < b$ where $f(a) \geq f(b)$. Then by the Mean Value Theorem, there exists $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \leq 0$$

   since the numerator is nonpositive and the denominator is negative. This contradicts the hypothesis.

(d) Suppose not, i.e. $\exists$ partition $\mathcal{P}$ of $[a,b]$ with $L(f;\mathcal{P}) > \int_a^b f$. Let $\epsilon = \frac{1}{2}\left(L(f;\mathcal{P}) - \int_a^b f\right) > 0$ and choose δ such that if $\mathcal{Q}$ is any tagged partition of $[a,b]$ with $||\mathcal{Q}|| < \delta$, then $|RS(f;\mathcal{Q}) - \int_a^b f| < \epsilon$, i.e. $RS(f;\mathcal{Q}) < \int_a^b + \epsilon < L(f;\mathcal{P})$. Now let $\mathcal{Q}$ be a common refinement of $\mathcal{P}$ and any other partition of $[a,b]$ with norm less than δ. We see $||\mathcal{Q}|| < \delta$ but since $\mathcal{Q} \geq \mathcal{P}$, we have

$$L(f;\mathcal{P}) \leq L(f;\mathcal{Q}) \leq RS(f;\mathcal{Q}) < \int_a^b f + \epsilon < L(f;\mathcal{P}).$$

This a contradiction (nothing is less than itself).

(e) Let $H(x) = \int_a^x (f-g)$. Since f and g are continuous, $f-g$ is continuous and H is therefore differentiable by the Fundamental Theorem of Calculus. Observe $H(a) = H(b) = 0$ so by Rolle's Theorem, there is $c \in (a,b)$ with $H'(c) = (f-g)(c) = 0$. Thus $f(c) = g(c)$.

(f) For each n, let s_n be the n^{th} partial sum of $\sum a_n$ and let t_n be the n^{th} partial sum of b_n. We see (since $0 \leq a_n$ and $0 \leq b_n$ for all n) that $\{s_n\}$ and $\{t_n\}$ are increasing sequences of nonnegative real numbers, and from the hypothesis we have $s_n \leq t_n$ for all n. We are given that $\sum b_n = \lim t_n$ exists; since $\{t_n\}$ is increasing we have $\lim t_n = \sup\{t_n\}$ so $s_n \leq t_n \leq \sum b_n$ for all n. Thus $\{s_n\}$ is an increasing sequence, bounded above, so this sequence converges by the Monotone Convergence Theorem. By definition, $\sum a_n$ converges.

6. (a) If $x = y$, both sides of the inequality are zero. Now assume $x \neq y$; WLOG $y < x$ (otherwise reverse x and y). By the Mean Value Theorem we have $c \in (y,x)$ such that

$$\frac{\cos x - \cos y}{x - y} = -\sin c;$$

taking absolute values of both sides of this equality we have

$$\frac{|\cos x - \cos y|}{|x-y|} = |-\sin c| \leq 1;$$

the result follows by multiplying through this inequality by $|x-y|$.

(b) Assume f is differentiable at x_0. Then, from result proved in class,

$$\lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} = f'(x_0).$$

Now

$$\begin{aligned}
\lim_{h\to 0} \frac{f(x_0+\alpha h) - f(x_0+\beta h)}{h} &= \lim_{h\to 0} \frac{f(x_0+\alpha h) - f(x_0) + f(x_0) - f(x_0+\beta h)}{h} \\
&= \lim_{h\to 0} \frac{f(x_0+\alpha h) - f(x_0)}{h} + \lim_{h\to 0} \frac{f(x_0) - f(x_0+\beta h)}{h} \\
&= \lim_{s\to 0} \frac{f(x_0+s) - f(x_0)}{\frac{s}{\alpha}} + \lim_{t\to 0} \frac{f(x_0) - f(x_0+t)}{\frac{t}{\beta}} \\
&\quad \text{(by setting } s = \alpha h \text{ in the first limit and setting} \\
&\quad\quad t = \beta h \text{ in the second limit)} \\
&= \alpha f'(x_0) - \beta f'(x_0) \\
&= (\alpha - \beta) f'(x_0).
\end{aligned}$$

(c) (Assume WLOG that $f(z) \neq g(z)$.) Let $\epsilon > 0$. Now choose (since f is Riemann integrable on $[a,b]$) $\eta > 0$ such that if $\mathcal{P}$ is any partition of $[a,b]$ with $||\mathcal{P}|| < \eta$, then $\left|RS(f;\mathcal{P}) - \int_a^b f\right| < \frac{\epsilon}{2}$. Let $\delta = \min(\eta, \frac{\epsilon}{4|g(z)-f(z)|})$ and let $\mathcal{P} = \{x_0, ..., x_n\}$ be any partition of $[a,b]$ with $||\mathcal{P}|| < \delta$. We have

$$\begin{aligned}\left|RS(g;P) - \int_a^b f\right| &\leq |RS(g;P) - RS(f;P)| + \left|RS(f;P) - \int_a^b f\right| \\ &< \left|\sum_{j=1}^n [g(c_j) - f(c_j)]\,\Delta x_j\right| + \frac{\epsilon}{2} \\ &\leq \sum_{j=1}^n |g(c_j) - f(c_j)|\,\Delta x_j + \frac{\epsilon}{2} \\ &= \sum_{\{j : z \in [x_{j-1}, x_j]\}} |g(c_j) - f(c_j)|\,\Delta x_j + \frac{\epsilon}{2} \\ &\quad \text{(since } f(x) = g(x) \text{ on all subintervals not containing } z\text{)} \\ &\leq 2\delta\,|g(z) - f(z)| + \frac{\epsilon}{2} \\ &\quad \text{(since there can be at most two subintervals containing } z\text{)} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\end{aligned}$$

so $g \in R([a,b])$ and $\int_a^b g = \int_a^b f$ by definition.

(d) By additivity, we see that for all $x \in [0,1]$,

$$\int_0^1 f(t)\,dt = \int_0^x f(t)\,dt + \int_x^1 f(t)\,dt = 2\int_0^x f(t)\,dt.$$

Treating the far-left and far-right sides of this equation as functions of x and differentiating both sides, we obtain $0 = 2f(x)$, i.e. $f(x) = 0$ for all x.

7. (a) By direct calculation, we see that for $x \neq 0$,

$$\frac{f(x) - f(0)}{x - 0} = x \sin\left(\frac{1}{x^2}\right).$$

Now observe that $-|x| \leq x\sin\left(\frac{1}{x^2}\right) \leq |x|$ for all x, so by the Squeeze Theorem,

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} x \sin\left(\frac{1}{x^2}\right) = 0.$$

(b) Consider the sequence $\{x_n\} \subseteq [0,1]$ defined by $x_n = \frac{1}{(2n+1)\pi}$. We have, for each n,

$$f'(x_n) = \frac{2}{(2n+1)\pi}\sin[(2n+1)\pi)] - 2(2n+1)\pi\cos[(2n+1)\pi] = 0 + 2(2n+1) = 4n+2$$

so $\{f'(x_n)\}$ is an unbounded sequence since it properly diverges to ∞. Thus f' is unbounded on $[0,1]$, hence not integrable on $[0,1]$, so $\int_0^1 f'(x)\,dx$ does not exist. Thus the statement is false.

8. (a) Let $\epsilon > 0$; choose $N_1 \in \mathbb{N}$ so that $n \geq N_2 \Rightarrow d(x_m, x_n) < \frac{\epsilon}{2}$ and choose $N_2 \in \mathbb{N}$ so that $n \geq N_2 \Rightarrow d(y_m, y_n) < \frac{\epsilon}{2}$. Set $N = \max(N_1, N_2)$ and suppose $m, n \geq N$. Then

$$d(x_n, y_n) \leq d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n) < \frac{\epsilon}{2} + d(x_m, y_m) + \frac{\epsilon}{2}$$

and

$$d(x_m, y_m) \leq d(x_m, x_n) + d(x_n, y_n) + d(y_n, y_m) < \frac{\epsilon}{2} + d(x_n, y_n) + \frac{\epsilon}{2};$$

the second inequality implies

$$d(x_n, y_n) \geq d(x_m, y_m) - \epsilon.$$

Putting the first and last inequalities together, we see $|d(x_m, y_m) - d(x_n, y_n)| < \epsilon$, so $\{d(x_n, y_n)\}$ is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete, $\{d(x_n, y_n)\}$ converges.

(b) Let $h : \mathbb{R} \to \mathbb{R}$ be defined by $h(x) = [f(x)]^2 - [f'(x)]^2$. Since f is twice-differentiable, h is differentiable with $h'(x) = 2f(x)f'(x) - 2f'(x)f''(x) = 2f(x)f'(x) - 2f'(x)f(x) = 0$ (applying the hypothesis) and therefore h is constant, in particular $h(x) = h(0)$ for all x. This means $[f(x)]^2 - [f'(x)]^2 = [f(0)]^2 - [f'(0)]^2 = 0$ for all x (using the hypothesis $|f(0)| = |f'(0)|$. Thus $[f(x)]^2 = [f'(x)]^2$ for all x; taking square roots of both sides yields the result.

(c) Let $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) = x^\alpha$; f is continuous on $[0, 1]$ (hence $f \in R([0, 1])$) and has antiderivative $F(x) = \frac{x^{\alpha+1}}{\alpha+1}$ so by the Fundamental Theorem of Calculus,

$$\int_0^1 f = F(1) - F(0) = \frac{1}{\alpha + 1}.$$

Let $\mathcal{P}_n = \{0, \frac{1}{n}, \frac{2}{n}, ..., 1\}$ be the partition of $[0, 1]$ into n equal length subintervals. We see that $||\mathcal{P}_n|| = \frac{1}{n} \to 0$ as $n \to \infty$, so by result from class $U(f; \mathcal{P}_n) \to \int_0^1 f = \frac{1}{\alpha+1}$. But since f is increasing, for each j we have

$$\sup \left\{ f(x) : x \in \left[\frac{j-1}{n}, \frac{j}{n} \right] \right\} = f\left(\frac{j}{n}\right) = \left(\frac{j}{n}\right)^\alpha$$

so

$$U(f; \mathcal{P}_n) = \sum_{j=1}^{n} \left(\frac{j}{n}\right)^\alpha \frac{1}{n} = \sum_{j=1}^{n} \frac{j^n}{n^{\alpha+1}} = \frac{1^\alpha + 2^\alpha + 3^\alpha + ... + n^\alpha}{n^{\alpha+1}}.$$

Therefore

$$\lim_{n\to\infty} \frac{1^\alpha + 2^\alpha + 3^\alpha + ... + n^\alpha}{n^{\alpha+1}} = \lim_{n\to\infty} U(f; \mathcal{P}_n) = \int_0^1 f = \frac{1}{\alpha + 1}.$$

(d) Note that for each n, $-|a_n| \leq a_n \leq |a_n|$. Therefore if we set r_n to the the n^{th} partial sum of the series $\sum -|a_n|$, set s_n to be the n^{th} partial sum of the series $\sum a_n$, and set t_n to be the n^{th} partial sum of the series $\sum |a_n|$, we observe that $r_n \leq 0 \leq t_n$ for all n, $\{r_n\}$ is a decreasing sequence converging to $-\sum |a_n| = \inf\{r_n\}$ and $\{t_n\}$ is an increasing sequence converging to $\sum |a_n| = \sup\{t_n\}$.

As $-\sum |a_n| \leq r_n \leq s_n \leq t_n \leq \sum |a_n|$ for all n, we see (by taking limits on the first, third and fifth terms of this inequality as $n \to \infty$) that $-\sum |a_n| \leq \lim s_n = \sum a_n \leq \sum |a_n|$. The result follows by definition of absolute value.