**Directions:** This exam has 6 questions, which are to be completed on your own time, and turned in to me no later than class time on Thursday, March 21. Please do each (numbered) problem on a separate piece of paper. I expect your solutions to be legible and well-organized (you will be graded mostly on mathematical correctness but also somewhat on style).

**Ground rules:** You may use your notes and the Rosenlicht and Gelbaum/Olmstead texts, but you may not use other textbooks or sources (including other people and the internet). You are not to work with others, but you may ask me for hints (the amount of help I am willing to give you varies from problem to problem).

When asked to prove a result, you may use any result from class, or any result proven in the homework with one exception: if asked to "reprove" a result that was stated or proven in class or in the homework, you cannot use that result itself.

If you have any questions about the ground rules, let me know.

**Questions:**

1. (a) (20 pts) For each $n \geq 1$, let $a_n = \frac{(-1)^n}{\sqrt{n}}$. Prove that $a_n \to 0$.

   (b) (20 pts) For each $n \geq 2$, let $a_n = n^{-n}$. Prove that $a_n \to 0$.
   *Hint:* The Squeeze Theorem may be helpful here.

2. (20 pts) For each $n \geq 0$, let $a_n = 3\cos\left(\frac{n\pi}{2}\right)$. Prove that the sequence $\{a_n\}$ diverges.

3. (20 pts) Let $S_1$ and $S_2$ be sets of real numbers, both of whom are bounded above. Prove that $S_1 \cup S_2$ is bounded above and prove that $\sup(S_1 \cup S_2) = \max(\sup S_1, \sup S_2)$.

4. (20 pts) Let $\{a_n\}$ and $\{b_n\}$ be Cauchy sequences of real numbers. Prove that $\{a_n + b_n\}$ is a Cauchy sequence.

   *Note:* This result was stated but not proven in class. You have to prove it.

5. (20 pts) Let $D$ be the set of real numbers in the interval $[0, 1]$ whose decimal expansions include only the digits 3 and 6. Determine, with proof, whether or not $D$ is countable.

6. Let $(X, d)$ be a metric space and let $A \subseteq X$. Define the *closure* of $A$ in $(X, d)$, denoted $\overline{A}$, to be the intersection of all closed subsets of $(X, d)$ containing $A$.

   (a) (5 pts) Prove that $\overline{A}$ is always closed.

   (b) (5 pts) Prove $A \subseteq \overline{A}$.

   (c) (5 pts) Prove $A = \overline{A}$ if and only if $A$ is closed.

   (d) (5 pts) Prove that $x \in \overline{A}$ if and only if for every $\epsilon > 0$, $B_\epsilon(x) \cap A \neq \emptyset$.

   (e) (5 pts) Describe (with proof) the set $\overline{\mathbb{Q}}$, the closure of $\mathbb{Q}$ in $(\mathbb{R}, d)$ (where $d$ the usual metric).

   (f) (5 pts) Describe (with proof) the set $\overline{\mathbb{Q}}$, the closure of $\mathbb{Q}$ in $(\mathbb{R}, d)$ (where $d$ is the discrete metric).