Directions: This exam has 6 questions, which are to be completed on your own time, and turned in to me no later than 3 PM on Tuesday, May 7 . Please do each (numbered) problem on a separate piece of paper. I expect your solutions to be legible and well-organized (you will be graded mostly on mathematical correctness but also somewhat on style).

Ground rules (same as Exam 1): You may use your notes and the Rosenlicht and Gelbaum/Olmstead texts, but you may not use other textbooks or sources (including other people and the internet). You are not to work with others, but you may ask me for hints.

When asked to prove a result, you may use any result from class, or any result proven in the homework with one exception: if asked to "reprove" a result that was stated or proven in class or in the homework, you cannot use that result itself.

1. (20 pts) Let (X, d) and (X', d') be two metric spaces. A function $f : X \to X'$ is called *Lipschitz* if there is a constant $c \in \mathbb{R}$ such that $d'(f(x), f(y)) < c \cdot d(x, y)$ for all $x, y \in X$. Prove that every Lipschitz function is continuous.

2. (20 pts) Prove, using the $\epsilon, \delta-$definition, the following limit statement:

$$\lim_{x \to 4} \frac{x^2 - 5x + 4}{3x - 12} = 1.$$

3. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

   (a) (20 pts) Prove that f is not continuous at any real number.
   Hint: Prove two things separately: first, that f is not continuous at any rational number; second, that f is not continuous at any irrational number.

   (b) (20 pts) Prove that there is no differentiable function $g : \mathbb{R} \to \mathbb{R}$ such that $g' = f$.
   Hint: A named theorem you proved in a homework set may be useful.

4. (20 pts) Suppose $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are differentiable functions such that $f(a) = g(a)$ and $f'(x) \leq g'(x)$ for all $x \geq a$. Prove that $f(b) \leq g(b)$ for all $b \geq a$.
   Hint: Apply the Mean Value Theorem to the function $g - f$.

5. (20 pts) Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be continuous functions on $[a, b]$ such that

$$\int_a^b f(x)\, dx = \int_a^b g(x)\, dx.$$

   Prove that there exists $c \in [a, b]$ such that $f(c) = g(c)$.
   Hint: Define $h : \mathbb{R} \to \mathbb{R}$ by $f(x) = \int_a^x [f(t) - g(t)]\, dt$; apply a named theorem to h.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x^2}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

   (a) (20 pts) Prove f is differentiable at 0, and calculate $f'(0)$.

   (b) (10 pts) By the Product and Chain Rules, for all $x \neq 0$ the derivative of f exists and is

$$f'(x) = 2x \sin\left(\frac{1}{x^2}\right) - \frac{2}{x} \cos\left(\frac{1}{x^2}\right).$$

   True or false (with proof): $\int_0^1 f'(x)\, dx = f(1) - f(0)$.