Combinatorics projects

for non-combinatorists

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March 31st, 2017

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So, I'm invoking a little known bylaw of the Michigan MAA that says the first speaker at a conference gets to rename the conference.

Conference In Honor of DJ's Third Birthday

Some background

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*: sadly the continued existence of our masters program in math is doubtful :(

So what do I do?

So a number of years ago I set out to find some problems that are interesting to me, but are still accessible and fun to undergrads and/or beginning grad students.

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I am in no way a trained combinatorialist. Fortunately, very basic combinatorics can be picked up pretty quickly, and I found some problems I think are pretty interesting and mostly wide open.

Catalan defined, but meaningless

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Obviously, we want to know what this is good for.

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We will focus on three, more or less.

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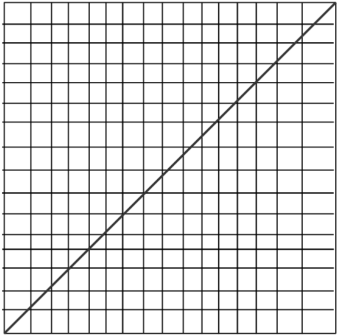

“Good” paths

We want to travel from the southwest corner to the northeast corner without crossing into the “bad” part of town, above the diagonal. (We can touch the diagonal, but not go above it.)

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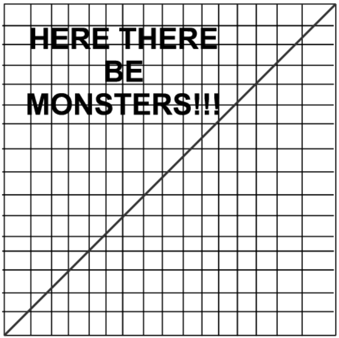

“Good” paths

Question - how many “good” paths are there?

Triangulations

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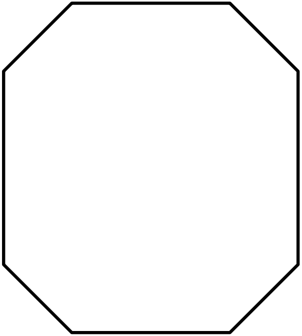

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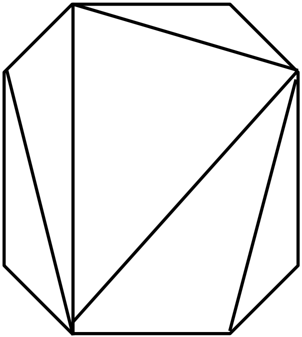

Triangulations

Question - how many triangulations of the $(n+2)$-gon are there?

Rooted binary trees

Consider a rooted binary tree:

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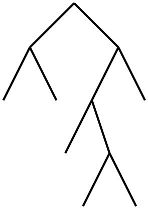

If a node has two descendants, we say that it is an *internal* node.

Rooted binary trees

Question - how many rooted binary trees are there with n internal nodes?

Ending the non-suspense

As you've probably guessed, the answer to all three of the above questions is c_n.

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Let's talk a little about why.

A clever idea

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We want to count b_n, all the rooted binary trees with n internal nodes. Except for now let's think about b_{n+1}.

We can partition the set of all binary trees with $n+1$ internal nodes according to how many internal nodes come from the left part of the top split, and how many come from the right part of the top split.

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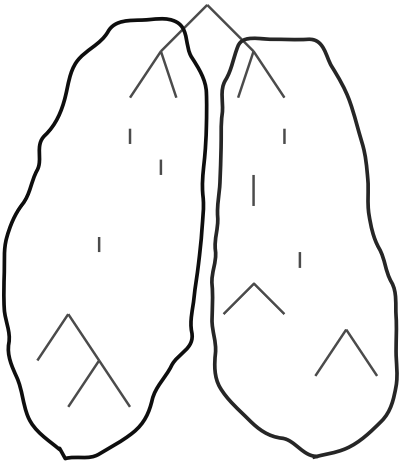

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And thus there are $b_k b_{n-k}$ binary trees with $n+1$ internal nodes with k on the left part and $n-k$ on the right.

Summing over all of the possible values of k, we get

$$b_{n+1} = \sum_{k=0}^{n} b_k b_{n-k}.$$

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Thus $xb(x)^2 + 1 = b(x)$.

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which gives us that $b_n = \frac{1}{n+1}\binom{2n}{n}$.

Beyond triangulations

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Theorem

(Hilton, Pedersen 1991) Fix p and k with $p \geq 3$. Write $n = (p-2)k + 2$. Then the number of ways to dissect an n-gon into p-gons is given by

$$c_{p,k} = \frac{1}{(p-2)k+1}\binom{(p-1)k}{k}.$$

Beyond triangulations

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Their proof was similar to the above, but some fancier complex analysis machinery was needed to create the generating function.

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In fact, there is a natural bijection between the dissections of the polygons and the trees, but we won't go into that here.

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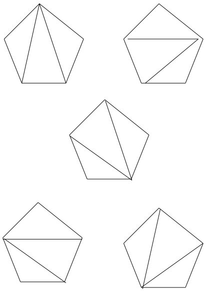

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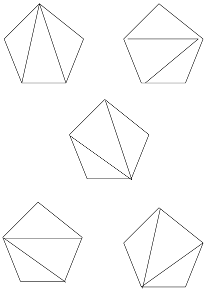

There are five of them, but they all seem "the same" in some sense.

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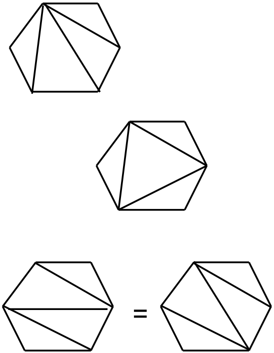

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To make this more precise, we may phrase it in terms of group actions:

Question: Note that the dihedral group of an $(n + 2)$-gon acts on the set of triangulations of an $(n + 2)$-gon. How many distinct orbits are there under this action?

Moon and Moser's answer

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(Here if k is not an integer then $c_k = 0$.)

Combining the above ideas

We can combine the above ideas: given positive integers p and k, and $n = (p-2)k + 2$, how many "distinct" dissections of an n-gon into p-gons are there (where here dissections are "distinct" if they are not equivalent via rotations or flips).

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...a messy formula too ugly to show in public.

Follow-up questions

(1) Can we get a nicer answer to the above question? We have a very ugly "mostly" closed form, but is there a cleaner answer?

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(2) If we can't get a cleaner formula, can we at least understand the asymptotics?

Back to trees

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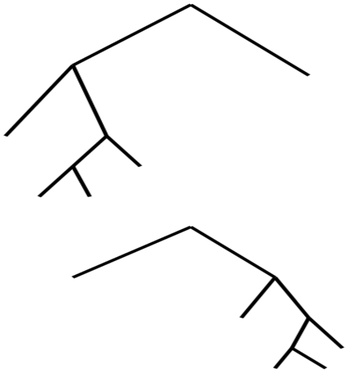
What do we mean by "different"? We say two binary trees are "equivalent" if we can apply a squence of "flips" at the internal nodes to transform one tree to another.

Picture of equivalent trees



For the group theorists out there

This seems to be an action of a wreath product group on the set of binary trees, using the Cauchy-Frobenius theorem. This should be pretty easy, I would think. So it should be done with a student.

"Different" p-ary trees

Similarly, we can ask how many "different" p-ary trees there are with a given number of internal nodes.

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It seems like the answer to this could be known somewhere, but it also seems it would be more fun to discover it with a student.

Tip of the iceberg

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These seem like a rich area for work with students, since there is little background material that is needed.

Intro to representation theory

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However, I managed to find a nice problem one can work on with students that doesn't require any actual knowledge of representations on their part.

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If your matrices have complex entries, we call these "ordinary" representations.

If your matrices have entries from a field of characteristic p, we call these "modular" representations.

Decomposing representations

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This is the basis of what is commonly called "block theory".

The symmetric groups

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But for the symmetric groups S_n, we have a number of advantages that we don't have for arbitrary finite groups. In particular, some easily defined combinatorics allows us to say things about the representations (even if we don't understand representation theory at all!)

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For instance, $\lambda = (4, 2, 2, 1)$ is a partition of 9.

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For instance, $\mu = (5, 4, 3, 3, 2, 2, 1)$ is a 3-regular partition of 20,

but $\nu = (5, 4, 3, 2, 2, 2, 1, 1)$ is *not* a 3-regular partition of 20.

Why symmetric groups are nice

Why should we care?

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The "important" modular representations of S_n are indexed by p-regular partitions of n.
All sorts of "important" information about these representations can be determined by doing some combinatorics to the partitions.

A very hard question

Given that the "important" representations of S_n are each indexed by certain classes of partitions, and one can in theory "decompose" ordinary representations of S_n (indexed by partitions of n) into "modular" representations of S_n (indexed by p-regular partitions of n), one might ask:

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Question: Given a partition λ of n and a prime p, how does the partition λ "decompose" into p-regular partitions?

For instance, for $n = 8$ and $p = 3$, we have the partition $(4, 2, 2)$ decomposes into the 3-regular partitions

$(7, 1)$, $(6, 2)$, $(4, 4)$, and $(4, 2, 2)$.

A very hard question

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Very, **very** hard. Whole books have been written about this for the symmetric group.

One way to organize this information (not to **compute** it, mind you, but organize it) is via the Brauer graph.

Brauer graphs of the symmetric group

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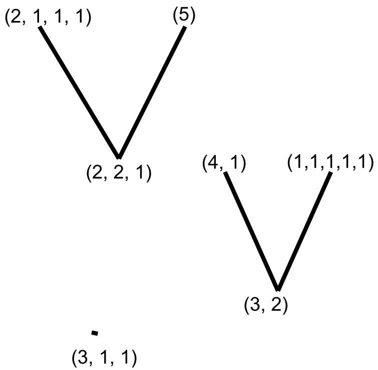
For our purposes, we define the Brauer graph of S_n for the prime p as follows:

The vertices of the graph are indexed by the partitions of n.

The vertices λ and σ share an edge if and only if the decompositions of λ and σ have a constituent in common.

Brauer graphs of the symmetric group

The n=5, p=3 Brauer graph:



Blocks for people who don't know/care about block theory

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Thus, in the above example, we have three blocks in our Brauer graph.

Our motivating question

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For instance, by way of comparison, it is known that if G is a group of odd order, any Brauer graph has diameter at most four.

Why this is hard

To get the Brauer graphs, we need to know how the partitions decompose, and to get how the partitions decompose, we need to know how the representations decompose.

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And it is really hard to say **anything** about how the representations decompose.

Some perspective

To give an idea of how hard this problem is in general, and to motivate our result, note that each block of a symmetric group comes equipped with the notion of a "weight".

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Blocks of weight $w = 1$ have a Brauer graph that is just a straight line of length p, thanks to some relatively fancy theory.

Blocks of weight $w = 2$ have only recently been characterized, and the nature of their Brauer graphs is still not well understood (this could be a good project!).

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Blocks of weight $w \geq 4$: we have no chance, really.

Rouquier blocks

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There are of course precise combinatorial definitions for all of this.

Why Rouquier blocks are nice

A very nice result of Chuang, Tan, and Turner (and independently by Leclerc and Miyachi) makes this hard problem in general quite nice for Rouquier blocks:

Theorem

Suppose R is a Rouquier block of S_n for some prime p, and let λ be a partition in R and μ be a p-regular partition in R. Then μ appears in the decomposition of λ if and only if ...

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(For those interested, the purely combinatorial statement involves quotients of partitions and Littlewood-Richardson coefficients)

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The point being, the above theorem translates the representation theoretic problem into a purely combinatorial problem, which can be (and has been!) tackled by undergrads who know nothing about representations (or even groups for that matter!).

Proof that this is accessible

Theorem

(Cossey, Mayer 2016) Let R be a Rouquier block for the prime p of weight w. Then the diameter of the Brauer graph of R is bounded below by $p-1$ and above by $p+2\lceil\log_2(w)\rceil+1$.

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We have shown these graphs are fairly close to complete - for instance, if $p = 11$ and $w = 10$, then our bound puts the diameter between 10 and 20, while the number of vertices is six or seven digits!

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(3) Can we find the maximally embedded complete graph?

(4) Any other graph theoretic question you might like about this graph.