Entropy of non rectangular LEGO bricks

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What is a LEGO Brick?

A *LEGO brick* is a plastic building toy which typically has studs on one side and holes on another side used for interlocking them.

Most LEGO bricks are rectangular prisms. Here is a picture of a 2×4 LEGO brick (the studs are on the top; the holes are on the bottom):

Question: Suppose you connect n LEGO bricks of the same size (and color) together. How many different buildings can you make?

Notation

Define B to be a specific type of LEGO brick (for example, a 2×4 brick).

Then let $T_B(n)$ be the number of buildings (counted up to rotations and translations) that can be constructed out of n bricks of type B.

Main Question: What kind of function is $T_B(n)$? How fast does it grow?

What is entropy?

Definition: The *entropy* of a LEGO brick of type B is the number

$$h_B = \lim_{n\to\infty} \frac{1}{n} \log T_B(n)$$

(that this limit exists needs to be proven).

Idea: The entropy of a function captures its exponential growth rate. If h_B exists and is finite, then $T_B(n) \sim 2^{h_B n}$ so T_B grows exponentially at rate h_B.

Note: we use log base 2, but the base is not important.

Remark: By "entropy", we mean information entropy, which is somewhat different than the thermodynamic entropy you learn about in chemistry.

History

In a paper published in 2014 by Durhuus and Eilers, the authors showed:

1. The entropy of any rectangular LEGO brick is finite.

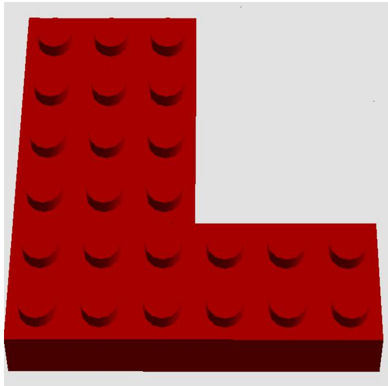
   (Reason: superadditivity of a sequence growing at the same rate as $\log T_B(n)$.)

2. $\log 78 \leq h_{2\times 4} \leq \log 177$. (The methods they use could be adapted to give bounds for any rectangular brick.)

We want to extend these results to other types of LEGO bricks.

L-shaped LEGO bricks

A brick in class $\mathcal{L}(B, W, b, w)$ is a $B \times W$ rectangular brick, with a $b \times w$ notch cut out of the upper-right corner (when the brick is rotated so that the side of length B is horizontal):



The picture above is a brick in class $\mathcal{L}(6, 6, 3, 4)$.

General results about L-shaped bricks

Lemma For any B, W, b and w,

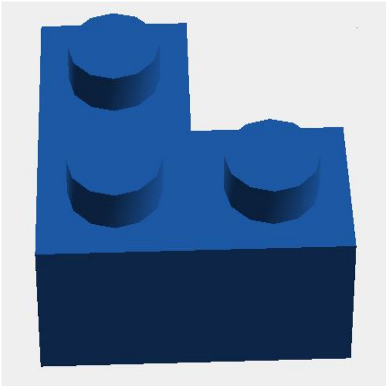
$$T_{\mathcal{L}(B,W,b,w)}(2) = 2(2B-1)(2W-1)+2(B+W-1)^2-9(B-b)(W-w).$$

Theorem 1 (McClendon-W) For any B, W, b and w, $h_{\mathcal{L}(B,W,b,w)}$ exists and is finite.

Theorem 2 (McClendon-W) $\log T_{\mathcal{L}(B,W,b,w)}(2) \le h_{\mathcal{L}(B,W,b,w)} \le$

$$\log\left(\frac{(2(BW-(B-b)(W-w))-1)^{BW-(B-b)(W-w)-1}(BW-(B-b)(W-w))}{(2(BW-(B-b)(W-w))-2)^{(BW-(B-b)(W-w))-2}}\right).$$

Our favorite example: $\mathcal{L}(2,2,1,1)$



From the formula on the previous slide:

$$T_{\mathcal{L}(2,2,1,1)}(2) = 27 \Rightarrow h_{\mathcal{L}(2,2,1,1)} \geq \log 27.$$

We have recently improved this lower bound to $\log 36$ and we think we can improve it further.

Our favorite example: $\mathcal{L}(2,2,1,1)$

Our best upper bound (as of now; we are planning to sharpen this) is

$$h_{\mathcal{L}(2,2,1,1)} \leq \log 146.$$

Where does this upper bound come from?

Finding the upper bound

Consider a finite string of $6(n-1)$ symbols taken from a "alphabet" of size 13.

Example: $0, 9, 0, 7, 0, 0, 0, 2, 0, 0, 6, 0, \ldots$

The first six symbols in this sequence tell us how to attach the next bricks to the first brick. The next six tell us how to attach to the second brick. The same can be done for this entire sequence of symbols.

Finding the upper bound

Some of these sequences will lead to contradictions; for example, if two bricks are forced to occupy the same space, or if the building is not contiguous.

The sequences that do not lead to a contradiction are called *allowable.* We can find an upper bound on the number of allowable sequences, giving us a upper bound on $T_B(n)$.