Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

Website: http://AIMsciences.org

pp. **X–XX**

AN AMBROSE-KAKUTANI REPRESENTATION THEOREM FOR COUNTABLE-TO-1 SEMIFLOWS

David M. McClendon

Northwestern University
2033 Sheridan Road
Evanston, IL 60208-2370, USA

Abstract. Let X be a Polish space and T_t a jointly Borel measurable action of $\mathbb{R}^+ = [0, \infty)$ on X by surjective maps preserving some Borel probability measure μ on X. We show that if each T_t is countable-to-1 and if T_t has the "discrete orbit branching property" (described in the introduction), then (X, T_t) is isomorphic to a "semiflow under a function".

1. Introduction.

1.1. Background.

In the 1940s Ambrose and Kakutani ([2], [3]) showed that any aperiodic, jointly measurable, measure-preserving flow on a standard probability space is measurably conjugate to a suspension flow (see section 1.2 for the definitions of "suspension flow" and other terms used here). Their result reveals connections between the dynamics of flows and the dynamics of induced maps and also serves as an jumping-off point for the theory of Kakutani equivalence [8] (two transformations are (measurably) *Kakutani equivalent* if there is a measure-preserving flow for which both transformations arise as return-time maps to measurable cross-sections of the flow). The Kakutani equivalence theory is quite rich and has led to greater understanding of more general notions of "equivalence" of measure-preserving systems (see [7], [16], [9]); further, it is intimately associated with the problem of classifying measure-preserving flows up to time-changes [7].

A purely descriptive set-theoretic notion of Kakutani equivalence was introduced by Nadkarni [15]; in a recent paper [14], Miller and Rosendal completely characterize this descriptive Kakutani equivalence for Borel automorphisms. In particular, they showed that in contrast to the rich theory of measurable Kakutani equivalence, all aperiodic, non-smooth Borel automorphisms are descriptive Kakutani equivalent (and consequently that all non-smooth free Borel flows are isomorphic up to a time-change).

In this paper we describe a class of semiflows which can be represented as suspension semiflows. Krengel [11] showed that any measure-preserving semiflow is a factor of a suspension semiflow. In the same paper he characterized in the suspension space the $\sigma-$algebra of sets which are lifts of measurable subsets of the original phase space under the factor map, thus obtaining a purely measurable representation of general semiflows. However, Krengel's measurable sets do not arise as (the

2000 *Mathematics Subject Classification.* Primary: 37A99; Secondary: 37A40,37B99.

Key words and phrases. Borel semiflow, measure-preserving semiflow, suspension semiflow.

completion of) the Borel sets with respect to any Polish topology on the suspension space (they do not separate points). For our class of semiflows, we obtain a topological as well as measure-theoretic picture.

It is hoped that this result will have many applications. First, in this paper we apply our result to show that semiflows of the class studied here have continuous versions, extending a theorem of Becker and Kechris about Polish group actions [4]. Second, it is hoped that the results here can be applied to give non-invertible versions of the Miller-Rosendal results; in particular we wish to obtain results classifying non-smooth free Borel semiflows up to a time-change.

Looking further, in [13], Lin and Rudolph described the notion of "Kakutani shift equivalence" for measure-preserving endomorphisms (this seems to be the natural generalization of measurable Kakutani equivalence for invertible transformations); it would be interesting to study the descriptive analogue of this notion.

1.2. **Definitions and main results.** Let $\widehat{X}$ be a Polish space (all Polish spaces in this paper are assumed to be uncountable) and let $\widehat{T} : \widehat{X} \to \widehat{X}$ be a Borel measurable map which preserves some Borel measure $\widehat{\mu}$ on $\widehat{X}$. Let f be a Borel measurable function from $\widehat{X}$ into $\mathbb{R}^+ = [0, \infty)$ with $\int_{\widehat{X}} f d\widehat{\mu} = 1$ and $\sum_{i=1}^{\infty} f(\widehat{T}^i \widehat{x}) = \infty \, \forall \widehat{x}$. Let X be the set $\{(x, t) \in \widehat{X} \times \mathbb{R}^+ : 0 \leq t < f(x)\}$ and define a measure μ on X by $\mu = \widehat{\mu} \times \lambda$ where λ is Lebesgue measure on $\mathbb{R}^+$. We define an action of $\mathbb{R}^+$ on X by

$$T_s(\widehat{x}, t) = \left(\widehat{T}^i(\widehat{x}), (s + t) - \sum_{j=0}^{i-1} f(\widehat{T}^j(\widehat{x})) \right)$$

where i is the largest integer such that $s + t \geq \sum_{j=0}^{i-1} f(\widehat{T}^j(\widehat{x}))$. The maps $\{T_s : s \in \mathbb{R}^+\}$ give a one-parameter family of Borel μ-preserving maps on X which we call a *suspension semiflow* or *semiflow under a function.* Under this action, a point $(\widehat{x}, t)$ flows vertically upward at unit speed until it hits the graph of f; the point then returns back to $\widehat{X}$ at the point $(\widehat{T}(\widehat{x}), 0)$ and continues upward. We call the function f the *return-time function* and call $\widehat{T}$ the *return-time map* or *induced map* for the suspension semiflow. $\widehat{X}$ is called the *base* of the semiflow.

If the map $\widehat{T}$ is invertible, then so is each T_s and we call the system a *suspension flow* or *flow under a function.* As mentioned earlier, Ambrose and Kakutani ([2], [3]) showed that any aperiodic jointly measurable flow by measure-preserving maps on a standard probability space $(X, \mathcal{F}, \mu)$ is measurably conjugate to a flow under a function where the base measure is finite. Wagh [18] carried out an analogue of the Ambrose-Kakutani result for measurable flows on standard Borel spaces, showing that any Borel flow is isomorphic to a suspension flow where the base is a Polish space and the return-time function is Borel. In the 1970s Rudolph [17] and Krengel [12] strengthened the Ambrose-Kakutani results; they showed that for any irrational $\alpha > 0$ and any $c \in (0, 1)$, any measure-preserving flow on a standard Lebesgue space is measurably conjugate to a flow under a function where the return-time function takes the value α on a set of measure c in the base and takes the value 1 on the rest of the base. In Section 2 we show that the Rudolph and Krengel results do not hold in general for suspension semiflows.

Definition 1.1. A *Borel semiflow* of $\mathbb{R}^+$ is a Polish space X with a probability measure μ defined on the $\sigma-$algebra $\mathcal{B}(X)$ of Borel subsets of X together with a collection of surjective maps $T_t : X \to X$ defined for each $t \in \mathbb{R}^+$ which satisfy

$T_0 = id_X$, $T_{s+t} = T_s \circ T_t$ for all s, t and for which the action is jointly Borel measurable, i.e. for every Borel $B \subseteq X$, the set $\{(x,t) \in X \times \mathbb{R}^+ : T_t(x) \in B\}$ is Borel, has no periodic points, and is measure-preserving, i.e. $\mu(T_{-t}(B)) = \mu(B)$ for every $B \in \mathcal{B}(X)$ and $t \geq 0$.

Remark 1. Any finite measure preserved by an aperiodic semiflow (or flow, for that matter) must be non-atomic, for if $\mu(\{x\}) > 0$, then $\mu(T_{-t}(x)) > 0$ for all $t > 0$ and hence $\mu(\bigcup_{t>0} T_{-t}(x)) = \infty$.

Definition 1.2. Two Borel semiflows $(X, \mathcal{B}(X), \mu, T_t)$ and $(Y, \mathcal{B}(Y), \nu, S_t)$ are *isomorphic* if there exists a Borel measurable bijection ϕ between (forward) invariant Borel subsets $X_0 \subseteq X$ and $Y_0 \subseteq Y$ of full measure ($\mu(X_0) = \nu(Y_0) = 1$) such that $\phi^*(\mu) = \nu$ and $\phi \circ T_t = S_t \circ \phi$ for all $x \in X_0$.

It is natural to ask which Borel semiflows are isomorphic to semiflows under a function. One easily sees that not every Borel semiflow on a Polish space can be conjugated to a suspension semiflow; given any suspension semiflow and any point not lying in the base $\widehat{X}$, there exists $t > 0$ such that $\#(T_{-t}(x)) = 1$, i.e. the semiflow is "invertible over a small period of time" for points not in the base. In particular this means that the set of points x such that $\forall\, t > 0$, $\#(T_{-t}(x)) > 1$ must occur only countably often along forward orbits, and the set of times at which any point enters this set must be discrete. We generalize this observation in the following definition:

Definition 1.3. Let (X, T_t) be a Borel semiflow. We say a point $x \in X$ has an *orbit branching* at time t_0 if

$$\bigcup_{t<t_0} T_{-t}T_t(x) \neq \bigcap_{t>t_0} T_{-t}T_t(x).$$

Given a point x, let $B(x) = \{t \geq 0 : x \text{ has an orbit branching at time } t\}$. A Borel semiflow is said to have the *discrete orbit branching property* if $B(x)$ is discrete for every point x.

Any suspension semiflow has the discrete orbit branching property, since for any point in such a system $B(x)$ is contained in the set of times $t \geq 0$ for which $T_t(x)$ lies in the base. The discrete orbit branching property, being preserved under isomorphism, is therefore a necessary condition for isomorphism with a suspension semiflow.

Example. Let X be the set of continuous, piecewise linear functions f from $\mathbb{R}^+$ to $\mathbb{R}^+$ which pass through the origin and for which there exists a $t_f \in (0, 1]$ such that on any interval taken from the collection $[0, t_f), (t_f, t_f + 1), (t_f + 1, t_f + 2), \ldots$, f has constant slope on that interval equal to either 0 or 1. Loosely speaking, we think of f as a solution to a "multi-valued differential equation" $f' \in \{0, 1\}$ with initial condition $f(0) = 0$. Define a semiflow T_t on X by $T_t(f)(s) = f(s + t) - f(t)$. This semiflow has discrete orbit branchings; $B(f) = t_f + \mathbb{Z}^+$ for any $f \in X$. (In fact, Theorem 1 of this paper guarantees this example is isomorphic to a suspension semiflow.)

Some models arising in economics [6] are similar to this example; they arise from "multi-valued differential equations" $f' \in \{f_1, \ldots, f_k\}$ where the f_j are smooth functions and some initial condition is specified. So long as the times where f' "changes" from f_i to f_j are suitably spread out, such a model will yield a semiflow like the one described above with discrete orbit branchings.

There is a second necessary condition as well: suppose for a given semiflow that there are two points $x \neq y \in X$ with $T_t(x) = T_t(y)$ for all $t > 0$. We say that x and y are *instantaneously and discontinuously identified* (IDI) by the semiflow. Suspension semiflows (as defined above) cannot have IDIs but they may occur in general Borel semiflows:

Example. Let $\sigma_2 : S^1 \to S^1$ defined by $\sigma_2 : x \mapsto 2x (\text{mod } 1)$. Let $Y = S^1 \times (0, 1]$ and consider a semiflow on Y which is like a suspension semiflow in that points flow upward at unit speed and return to the base via the return-time map σ_2. This is a Borel semiflow and can be thought of as a "closed on the top suspension" instead of the "closed on the bottom" suspension previously described. Points of the form $(a, 1)$ and $(a + 1/2, 1)$ are IDI by this semiflow. In particular, no invariant set for the semiflow can contain no points which are IDI.

We make the following definitions to describe this phenomenon: first the sets

$$IDI(T_t) = \{x \in X : \exists y \neq x \text{ such that } T_t(x) = T_t(y) \, \forall \, t > 0\}$$

$$I(x) = \{y \in X : T_t(x) = T_t(y) \, \forall \, t > 0\}$$

and also the equivalence relation

$$\mathsf{IDI} = \{(x, y) \in X \times X : T_t(x) = T_t(y) \, \forall \, t > 0\}.$$

Lemma 1.4. IDI *is a Borel subset of* $X \times X$.

Proof. Consider the Borel action $(X \times X, (T \times T)_t)$ of $\mathbb{R}^+$ defined by $(T \times T)_t(x, y) = (T_t(x), T_t(y))$. Denote the diagonal of $X \times X$ by Δ. Now

$$\begin{aligned} (x, y) \in \mathsf{IDI} \quad &\Leftrightarrow \quad (T \times T)_t(x, y) \in \Delta \, \forall \, t > 0 \\ &\Leftrightarrow \quad (x, y) \in \bigcap_{q \in \mathbb{Q} \bigcap (0, \infty)} (T \times T)_{-q}(\Delta) \end{aligned}$$

so IDI is a Borel relation as desired. □

To account for the possibility of IDIs, we introduce the notion of "suspension semiflow with IDIs". To describe this setup, start with an "ordinary" suspension semiflow with base G_1 and return-time function g. Add another space G_2 and replace the return-time map from G_1 to itself with a map taking G_1 to $G_1 \bigcup G_2$ and take a "projection" σ which maps G_2 into G_1 and is the identity on G_1. Under this construction, points in G_1 flow upward until they hit the graph of g, then return to G_1 or G_2. If they return to G_1, they again flow upward as usual. If they return to some $x \in G_2$, they flow upward through points sitting above some point $\sigma(x)$ in G_1 (in particular, for $x \in G_2, x$ and $\sigma(x)$ are IDI by this semiflow). More precisely:

Definition 1.5. Consider two Polish spaces G_1 and G_2 and a Borel measurable map $\widehat{S} : G_1 \to G_1 \bigcup G_2$ which preserves some $\sigma-$finite Borel measure $\widehat{\mu}$ on $G_1 \bigcup G_2$ for which $\widehat{\mu}(G_2) = 0$. Let $\sigma : G_1 \bigcup G_2 \to G_1$ be a Borel map which is the identity when restricted to G_1. Let $g : G_1 \to \mathbb{R}^+$ be Borel measurable with $\int_{G_1} g \, d\widehat{\mu} = 1$.

A *suspension semiflow with IDI* consists of the space

$$G = \left\{(x, t) \in G_1 \times \mathbb{R}^+ : 0 \leq t < g(x)\right\} \bigcup \left(G_2 \times \{0\}\right),$$

endowed with the Borel structure inherited from the product topology on $(G_1 \bigcup G_2) \times \mathbb{R}^+$ and the measure $\widehat{\mu} \times \lambda$, and the Borel semiflow S_s on G defined as follows:

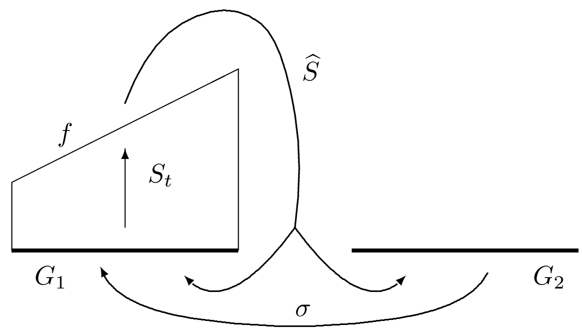

FIGURE 1. A suspension semiflow with IDI. Points flow upward through $G_1 \times \mathbb{R}^+$ until they hit the graph of f, then return to either G_1 or G_2 via $\widehat{S}$. Every point z in G_2 is IDI with some $\sigma(z)$ in G_1.

first let i be the largest integer such that $(s+t) \geq \sum_{j=0}^{i-1} g(\sigma(\widehat{S}^j(\widehat{x}))$; if $(s+t) > \sum_{j=0}^{i-1} g(\sigma(\widehat{S}^j(\widehat{x}))$ we let

$$S_s(x,t) = \left(\sigma(\widehat{S}^i(x)), (s+t) - \sum_{j=0}^{i-1} g(\sigma(\widehat{S}^j(\widehat{x}))\right)$$

and if $(s+t) = \sum_{j=0}^{i-1} g(\sigma(\widehat{S}^j(\widehat{x}))$ then

$$S_s(x,t) = \left(\widehat{S} \circ (\sigma \circ \widehat{S})^{i-1}(x), 0\right).$$

Remark 2. It is sufficient in the above definition that G_1 and G_2 be Borel subsets of a standard Polish space X and $\widehat{S} : G_1 \to G_1 \bigcup G_2$ and $\sigma : G_1 \bigcup G_2 \to G_1$ be Borel measurable with respect to the Borel structure of X. Let $\mathcal{T}$ be the given topology on X; one can choose a Polish topology $\mathcal{T}' \supseteq \mathcal{T}$ on X for which G_1 and G_2 are clopen but $\mathcal{T}$ and $\mathcal{T}'$ have the same Borel sets. G_1 and G_2 are Polish under the relative $\mathcal{T}'-$topology.

A Borel semiflow is called *countable-to-1* if there is a forward-invariant Borel subset X_0 of the phase space with $\mu(X_0) = 1$ such that for all $x \in X_0$ and all $t \geq 0$, $T_{-t}(x)$ is countable. Since our notion of isomorphism only requires that a conjugacy between systems is defined on invariant sets of full measure, we can choose a Polish topology $\mathcal{T}'$ on X giving rise to the same Borel sets as the original topology such that X_0 is a G_δ with respect to $\mathcal{T}'$. Then the restriction of T_t to X_0 is a Borel semiflow such that for every point x and every time $t \geq 0$, $T_{-t}(x)$ is countable. In what follows, we are thinking of this X_0 as X and assuming that every point has a countable number of preimages under each T_t.

Theorem 1.6. *Any countable-to-1 Borel semiflow $(X, \mathcal{B}(X), \mu)$ is isomorphic to a suspension semiflow with IDIs if and only if it has the discrete orbit branching property.*

Furthermore we have the following special case: when the the semiflow has no IDIs, we can choose $G_2 = \emptyset$ and obtain:

Theorem 1.7. *Suppose T_t is a countable-to-1 Borel semiflow with the discrete orbit branching property that has no IDIs (i.e. $T_t(x) = T_t(y) \,\forall\, t > 0$ implies $x = y$), then T_t is isomorphic to a suspension semiflow.*

Notice that in our definition of suspension semiflow we do not necessarily have a finite measure on the base $G_1 \bigcup G_2$. In Section 2 we give an explicit example of a Borel semiflow satisfying the hypotheses of Theorem 1.7 which cannot be isomorphic to a suspension semiflow whose base is endowed with a finite Borel measure. However, under an additional assumption we can conjugate a Borel semiflow to a suspension semiflow whose base measure $\widehat{\mu}$ is finite:

Theorem 1.8. *Given a countable-to-1 Borel semiflow T_t defined on a standard Polish space $(X, \mathcal{B}(X), \mu)$, if there exists a constant $c_0 > 0$ such that for every point $x \in X$, and every t_1 and t_2 in $B(x)$, we have $|t_1 - t_2| \geq c_0$, then $(X, \mathcal{B}(X), \mu, T_t)$ is isomorphic to a suspension semiflow with IDIs where the base measure $\widehat{\mu}$ is finite.*

We prove Theorems 1.6 and 1.8 in the next section. We first describe how to locate the orbit branchings of the semiflow via a countable list of Borel functions whose domains are Borel subsets of X. Then we take an appropriate cross-section for the semiflow (called "horizontally identifying") and construct the sets G_1 and G_2, the map $\widehat{S}$ and the function g needed to define a suspension semiflow with IDIs isomorphic to the original action.

We also show in the next section that given any countable-to-1 Borel semiflow satisfying the hypotheses of Theorem 1.7, one can choose a Polish topology on the phase space so that the action is jointly continuous, providing an extension of a theorem of Becker and Kechris [4] about Polish group actions to a class of actions of $\mathbb{R}^+$.

The following notation is used: Suppose (X, T_t) is a semiflow; for a point x we let $T_{-t}(x) = \{y : T_t(y) = x\}$; given a set $A \subseteq \mathbb{R}^+$, let $T_A(x) = \bigcup_{t \in A} T_t(x)$ and $T_{-A}(x) = \{y : \exists\, t \in A$ such that $T_t(y) = x\}$. Given a set $F \subseteq X$, $T_{-t}(F) = \{y : T_t(y) \in F\}$ and for $A \subseteq \mathbb{R}^+$, let $T_A(F) = \bigcup_{t \in A, x \in F} T_t(x)$ and $T_{-A}(F) = \{y : \exists\, t \in A$ such that $T_t(y) \in F\}$.

Throughout the sequel, X is assumed to be an uncountable Polish space with given topology $\mathcal{T}$ and T_t is assumed to be a countable-to-1 Borel semiflow satisfying the hypotheses of Theorem 1.

Remark 3. Countable-to-1 Borel semiflows have two key properties which are used in the proof of Theorem 1.6. First, since each T_t is countable-to-1, each T_t must send Borel sets to Borel sets (i.e. $T_t(A)$ is Borel for any Borel A and any $t \geq 0$). We will use this fact to show that the orbit branchings form a Borel set. Second, each $I(x)$ is at most countable. This ensures that the Borel equivalence relation **IDI** has a Borel transversal U (i.e. U intersects each **IDI**-equivalence class in exactly one point) and that there is a countable-to-1 Borel map σ mapping each $x \in X$ to its **IDI**-equivalent point in U. This σ is precisely the map σ described in the definition of a suspension semiflow with IDIs.

2. The proofs of Theorems 1.6 and 1.8.

2.1. Capturing the orbit branchings with Borel graphs.

We begin the proof of the theorem by describing how the orbit branchings can be covered by a countable list of Borel graphs. First, since each T_t is a Borel map from X to itself, for each

$q \in \mathbb{Q}^+ = \mathbb{Q} \bigcap [0,\infty)$ there is a Polish topology $\mathcal{T}_q \supseteq \mathcal{T}$ on X such that T_q is $\mathcal{T}_q-$continuous but $\mathcal{T}$ and $\mathcal{T}_q$ have the same Borel sets (see Theorem 13.11 of [10]).

Let $\{U_n\}$ be a countable basis for the topology $\mathcal{T}_q$. Now a point $x \in X$ has an orbit branching at time t if and only if for some n, we have

1. for all $q \in \mathbb{Q} \bigcap [0,t)$, $T_q(x) \notin T_q(U_n)$ and
2. for all $q \in \mathbb{Q} \bigcap (t,\infty)$, $T_q(x) \in T_q(U_n)$.

Since the semiflow is countable-to-1, all the $T_q(U_n)$ are Borel, so the set of pairs (x,t) such that x has an orbit branching at time t is a Borel subset of $X \times [0,\infty)$. Therefore by the Lusin-Novikov theorem, we can find a countable list of Borel functions $j_i : X \to [0,\infty)$ whose domains J_i are Borel subsets of X such that given any point $x \in X$, the orbit branchings of x occur at the times $\{j_i(x) : 1 \leq i < \infty\}$.

The functions thus constructed also satisfy the following key property:

Lemma 2.1. *Suppose x and y are two points in X with $T_t(x) = T_t(y)$. Then if we define $M(x) = \max\{s \leq t : s = j_i(x)$ for some $i\}$ (set $M(x) = \infty$ if no such s exists), then either $x = y$ or $M(x) = M(y)$.*

Proof. Suppose $x \neq y$. Then let $q = \sup\{s : T_s(x) \neq T_s(y)\}$; we have $0 \leq q \leq t$. Then $B(x) \bigcap [q,\infty) = B(y) \bigcap [q,\infty)$ so if $M(x) \geq q$, we are done. But certainly x has an orbit branching at time q so $q = j_i(x)$ for some i and hence $M(x) \geq q$. □

Remark 4. If we did not assume the semiflow was countable-to-1, we could still show using the above argument that the orbit branchings were an analytic subset of $X \times \mathbb{R}^+$ with discrete sections. Using a result of Lusin, one could then cover the orbit branchings with a countable list of Borel functions. However, those functions might not satisfy the preceding lemma, which is of importance with regard to what follows.

2.2. **Horizontally-identifying cross sections.** The existence of cross-sections for semiflows was first established by Krengel ([11] , [13]); we repeat some definitions here for convenience. Given a Borel semiflow $(X, \mathcal{F}, \mu, T_t)$, a set $F_0' \in \mathcal{F}$ is called a *thick section* for the semiflow if there exist parameters $0 < \alpha < \beta$ and a measurable function $\gamma : X \to [0,\infty)$ such that for every $x \in X$,

1. $\gamma(T_{\gamma(x)}x) \geq \beta$,
2. $\{T_t(x)\}_{\gamma(x) \leq t < \gamma(x)+\alpha} \subseteq F_0'$, and
3. $\{T_t(x)\}_{\gamma(x)+\alpha \leq t < \gamma(x)+\gamma(T_{\gamma(x)}(x))} \bigcap F_0' = \emptyset$.

Then a set F_0 is called a *section* or *cross-section* for the semiflow if F_0 consists of the left-endpoints of intervals of occurrence of some thick section F_0' on T_t-orbits. F_0 is endowed with a $\sigma-$algebra $\mathcal{G}_0$ of measurable sets as follows: a set $A \subseteq F_0$ belongs to $\mathcal{G}_0$ if $\overline{A} = \{T_t(x) : x \in A, 0 \leq t < \alpha\}$ is Borel. These are in our case precisely the Borel subsets of F_0. Given any section F_0, there is a return-time function $f(x) = \inf\{t \in \mathbb{R}^+ : T_t(x) \in F_0\}$; Lin and Rudolph [13] show that this map is $\mathcal{G}_0-$measurable and that one can choose a section so that the return-time function f is bounded above by some constant and below by some other positive constant. If this is the case we say that the section is *bounded*.

Call a section F_0 *horizontally identifying* if it is bounded and has the following two properties:

1. Whenever $T_{f(x)}(x) = T_{f(y)}(y)$ for some pair $x, y \in F_0$, then $f(x) = f(y)$.
2. Given any $x \in F_0$ and $t < f(x)$, the set $T_{-t}T_t(x)$ is contained in F_0.

Proposition 2.2. *Let F be a horizontally identifying section. For each pair of points $x, y \in F$, define*

$$A_y(x) = \{t \in [0, f(x)) : T_t(x) = T_s(y) \text{ for some } s \in [0, f(y))\}.$$

Then for all $x, y \in F, A_y(x) = A_x(y)$.

Proof. If $A_y(x)$ is empty, then the forward orbits of x and y are disjoint at least until they return to the section so $A_x(y)$ is also empty. Now if $t \in A_y(x)$, then there exists $s < f(y)$ so that $T_s(y) = T_t(x)$. By applying T to both sides, we get $T_{f(x)+s-t}(y) = T_{f(x)}(x)$ and $T_{f(y)}(y) = T_{f(y)+t-s}(x)$. But if the forward orbits of x and y meet before they return to the section, then they must coincide when they return to the section, i.e. $T_{f(x)}(x) = T_{f(y)}(y)$. Since F is horizontally identifying, $f(x) = f(y)$ so we have $T_{f(y)+s-t}(y) = T_{f(y)}(y)$ and $T_{f(x)+t-s}(x) = T_{f(x)}(x)$. If $s \neq t$, one of x or y must hit the section before its return time. This is impossible so $s = t$ and $t \in A_x(y)$. Thus $A_y(x) \subseteq A_x(y)$; by symmetry these sets must therefore coincide. □

This proposition ensures that for a horizontally identifying section F_0, any points which get identified before they return to the base must be identified at the same height, i.e. we cannot see points x, y in the base with $T_s(x) = T_t(y)$ (for $0 \leq s < f(x), 0 \leq t < f(y)$) but $s \neq t$. Also, if F_0 is a good section then for any point x in X there is a nonnegative number $\overline{f}(x)$ such that $T_{-\overline{f}(x)}(x) \subseteq F_0$ but $T_{-\alpha}(x) \bigcap F_0 = \emptyset$ for all $\alpha \in (0, \overline{f}(x))$ (recall that T_t is assumed to be surjective). So if one writes $x \in X$ as $T_t(y)$ where $y \in F_0$ and $0 \leq t < f(y)$, there is only one possible choice for t (of course there may be lots of choices for y).

Proposition 2.3. *Every Borel semiflow has a horizontally identifying cross-section F_0.*

Proof. From Lin and Rudolph [13], we can choose a Borel section F with return time bounded by some constant less than B. Then let $F_0 = T_{-B}(F)$. That F_0 is a section is obvious from the definition. Let f be the return time function for F_0; clearly $f < B$ so F_0 is a bounded section. Now suppose $x, y \in F_0$ are such that $T_{f(x)}(x) = T_{f(y)}(y)$. Assume $f(x) \geq f(y)$, and let $\delta = f(x) - f(y)$. Now $T_B(x)$ and $T_B(y) = T_{B+\delta}(x)$ are both in F. So $T_\delta(x) \in F_0$ and as $0 \leq \delta < f(x)$, $\delta = 0$.

Finally, to verify the second condition in the definition of horizontally identifying section, take $x \in F_0$ and $t < f(x)$ and consider some $y \in T_{-t}T_t(x)$. Then $z = T_t(x) = T_t(y)$ satisfies $T_{B-t}(x) \in F$ so $y \in F_0$ as desired. Thus F_0 is horizontally identifying. □

2.3. **Constructing the suspension.** Let F_0 be a horizontally identifying section for $(X, \mathcal{F}, \mu, T_t)$ with return-time function $f : F_0 \to [b, B]$ $(b > 0)$. Denote the Poincaré return map from F_0 to itself by $\widehat{T}$; remembering that F_0 is a subset of X we have $\widehat{T}(y) = T_{f(y)}(y)$ for any $y \in F_0$. Finally define

$$F = \{(x, t) \in F_0 \times \mathbb{R}^+ : 0 \leq t < f(x)\}$$

and notice that the map $\Upsilon : F \to X$ defined by $\Upsilon(x, t) = T_t(x)$ is measurable.

Lemma 2.4. *Suppose (x_1, t_1) and (x_2, t_2) are two points in F with $T_{t_1}(x_1) = T_{t_2}(x_2)$ (this implies that $t := t_1 = t_2$ since F_0 is horizontally identifying). Then:*

1. *$B(x_1) \bigcap [0, t]$ and $B(x_2) \bigcap [0, t]$ are nonempty,*
2. *$B(x_1) \bigcap [t, \infty) = B(x_2) \bigcap [t, \infty)$, and*

3. $\max\{s \leq t : s \in B(x_1)\} = \max\{s \leq t : s \in B(x_2)\}$

Proof. Statements (1) and (2) follow directly from the definition of orbit branching. For the third statement, first observe that if $t \in B(x_1)$ we are done. Otherwise, by the discrete orbit branching property there is some $s \in (0,t)$ such that $B(x_1) \bigcap (s,t] = \emptyset$. Then for every r in $[0, t-s)$, we have $\#(T_{-r}T_t(x)) = 1$. But $T_{t-r}(x_1)$ and $T_{t-r}(x_2)$ both belong to $T_{-r}T_t(x)$ so they are therefore equal. Then by applying statement (2) of this Lemma, we have

$$B(x_1) \bigcap [t-r, \infty) = B(x_2) \bigcap [t-r, \infty)$$

whence $B(x_1) \bigcap (s, \infty) = B(x_2) \bigcap (s, \infty)$. This implies $B(x_2) \bigcap (s,t] = \emptyset$ and we have $\max\{s \leq t : s \in B(x_1)\} \geq \max\{s \leq t : s \in B(x_2)\}$. By symmetry these quantities must be equal. □

Define the function $k : F \to \mathbb{R}^+$ by

$$k(x,t) = \begin{cases} t - \max\{s : s \leq t \text{ and } s \in B(x)\} & \text{if } B(x) \bigcap [0,t] \neq \emptyset \\ t & \text{if } B(x) \bigcap [0,t] = \emptyset \end{cases}$$

Lemma 2.5. *$k : F \to \mathbb{R}^+$ is Borel measurable.*

Proof. Observe that $((x,t),y) \in F \times \mathbb{R}^+$ lies on the graph of k if and only if either (1) $y = t$ and $j_i(x) > t$ for all i such that x lies in the domain of j_i or (2) there is some positive integer N such that

$$j_N(x) = t - y \text{ and } j_n(x) > t \forall n \geq N.$$

Therefore the graph of k is Borel and consequently k is itself Borel. □

In particular, Lemma 2.4 tells us $k(x_1,t_1) = k(x_2,t_2)$ whenever $\Upsilon(x_1,t_1) = \Upsilon(x_2,t_2)$. Therefore k passes to a well-defined Borel measurable function on X as well.

Recall that since T_t is countable-to-1, there is a Borel transversal U for **IDI** and a Borel function σ defined on X by setting $\sigma(x) \in U \bigcap$ **IDI**.

Lemma 2.6. *Suppose x_1 and x_2 satisfy $T_{k(x)}(x_1) = T_{k(x)}(x_2) = x$. Then $\sigma(x_1) = \sigma(x_2)$.*

Proof. Again write $x = T_t(y)$ with $y \in F_0$ and $0 \leq t < f(y)$. Assume $x_1 \neq x_2$; otherwise we are done. Let A be the set of times $t \geq 0$ for which $T_t(x_1) \neq T_t(x_2)$; $0 \in A$ so A is nonempty. Let $s = \sup\{t : t \in A\}$; x_1 has an orbit branching at time s so y has orbit branching at time $t - k(x) + s$. This is only possible if $s = 0$; i.e. $A = \{0\}$. Thus $(x_1, x_2) \in$ **IDI** so $\sigma(x_1) = \sigma(x_2)$. □

For $x \in X$, let $r_1(x) = \inf\{t > 0 : T_t(x) \in F_0\}$. Now let

$$g(x) = \min\{r_1(x), j_1(x), j_2(x), j_3(x), \ldots\}.$$

$g : X \to \mathbb{R}^+$ is Borel. Next let $G_0 = \{T_{-k(x)}x : x \in X\}$; this is a Borel subset of X.

Lemma 2.7. *$x \in G_0$ if and only if $k(x) = 0$ if and only if $x = T_{g(y)}(y)$ for some $y \in X$.*

Proof. Suppose $x \in G_0$; write $x = T_t(z)$ where $z \in F_0$ and $0 \leq t < f(z)$. Then since $x \in G_0$, $t \in B(z)$ so $k(z,t) = k(x) = 0$. Also, since $B(z)$ is discrete one can choose $s < t$ such that $B(z) \bigcap [s,t) = \emptyset$. Let $y = T_s(z)$; by Lemma 2.4 we conclude $g(y) = t - s$ so $x = T_{g(y)}(y)$. The other implications are immediate from the definitions. □

Let S_t be the suspension semiflow with IDI defined as in the introduction with $G_1 := G_0 \bigcap U$ and $G_2 := G_0 - U$ where σ is as described in Section 1 and $\widehat{S}(x) = T_{g(x)}(x)$. In particular the phase space G for this suspension semiflow is the subset of $X \times \mathbb{R}^+$ defined by

$$\left\{(x,t) \in (G_0 \bigcap U) \times \mathbb{R}^+ : 0 \le t < g(x)\right\} \bigcup \left((G_0 - U) \times \{0\}\right).$$

Lemma 2.6 assures us that the expression $\sigma(T_{-k(x))}(x)$ is a well-defined expression depending only on x. Let $\Sigma(x) = \sigma(T_{-k(x)}(x))$.

Lemma 2.8. *Let $\Sigma : X \to T \subset X$ be defined as above. Σ is Borel measurable and $\Sigma(X) \subseteq G_0$.*

Proof. The first statement is obvious because Σ is the composition of Borel maps. To prove the second statement, it suffices to verify $\Sigma(G_0) \subseteq G_0$. First observe $\Sigma|_{G_0} = \sigma|_{G_0}$ and since σ is the identity on T it is enough to show $\sigma(x) \in G_0$ for $x \in G_0 - T$. If $x \notin T$ then $(x, y) \in$ **IDI** for some $y = \sigma(x) \in T$. As y has orbit branching at time zero, $k(y) = 0$ so $y \in G_0$ as well. □

Now we define $\phi : X \to G$ by

$$\phi(x) = \begin{cases} (x, 0) & \text{if } x \in G_0 \\ (\Sigma(x), k(x)) & \text{if } x \notin G_0 \end{cases}.$$

Observe that ϕ is Borel since Σ and k are Borel functions; we will see that ϕ is an isomorphism between (X, T_t) and (G, S_t). One can check that ϕ is bijective with inverse given by $(x, t) \mapsto T_t(x)$.

Lemma 2.9. *Given any $x \in X$ and any $t \ge 0$, we have $\phi \circ T_t(x) = S_t \circ \phi(x)$.*

Proof. We consider four cases:

Case 1: Suppose first that $x \in G_0$ and $t \in [0, g(\sigma(x)))$. In this case we have $k(T_t(x)) = t$ and $\Sigma(T_t(x)) = \sigma(x)$ by definition and therefore $\phi(T_t(x)) = (\sigma(x), t) = S_t(\phi(x))$.

Case 2: If $x \in G_0$ and $t = g(\sigma(x))$ (this means $t > 0$ and $T_t(\sigma(x)) = T_t(x)$) then

$$S_t(\phi(x)) = (\widehat{S}(\sigma(x)), 0) = (T_{g(\sigma(x))}(\sigma(x)), 0) = (T_t(x), 0) = \phi(T_t(x)).$$

Case 3: If $x \notin G_0$, then $\phi(x) = (\Sigma(x), k(x))$ where $k(x) > 0$. Suppose that $0 \le t < g(\Sigma(x)) - k(x)$. In this case, $\phi(T_t(x)) = (\Sigma(T_t(x)), k(T_t(x))) = (\Sigma(x), k(x) + t) = S_t(\phi(x))$.

Case 4: If $x \notin G_0$, again write $\phi(x) = (\Sigma(x), k(x))$ where $k(x) > 0$; we suppose now that $t = g(\Sigma(x)) - k(x)$. Here we see

$$\begin{aligned} S_t(\phi(x)) &= (\widehat{S}(\sigma(T_{-k(x)}(x))), 0) \\ &= (T_{g(\Sigma(x))}(\sigma(T_{-k(x)}(x))), 0) \\ &= (T_{g(\Sigma(x)) - k(x)}(x), 0) \\ &= (T_t(x), 0) \\ &= \phi(T_t(x)). \end{aligned}$$

The preceding four cases are sufficient because for every x, the set of times t where $T_t(x) \in G_0$ tends to ∞; therefore one can write $t = t_1 + t_2 + ... + t_n$ appropriately so that the cases above can be applied for each t_i. □

At this point we have constructed an isomorphism ϕ between $(X, \mathcal{B}(X), \mu, T_t)$ and $(G, \mathcal{B}(G), \phi^*(\mu), S_t)$. It remains to show that $\phi^*(\mu)$ is the product of a $\sigma-$finite measure $\widehat{\nu}$ on G_0 (assigning mass 1 to $G_0 \bigcap U$) with Lebesgue measure in the vertical direction.

We begin by recalling from Krengel and Lin/Rudolph that we can place a Borel probability measure $\overline{\mu}$ on F_0 as follows: pick c such that $f(y) > c$ for all $y \in F_0$; then given a Borel $A \subseteq F_0$, define

$$\overline{\mu}(A) = \frac{\mu(T_{[0,c)}(A))}{\mu(T_{[0,c)}(F_0))}.$$

In particular, we know that for any Borel B in F satisfying the following property:

$$(y,t) \in B \text{ and } T_t(y) = T_t(z) \Rightarrow (z,t) \in B \qquad (1)$$

we have $(\overline{\mu} \times \lambda)(B) = \mu(\Upsilon(B))$.

For each $i > 0$, place a finite Borel measure ν_i on the set $G(i) = \{(y,t) \in F : j_i(y) = t \text{ and } t > 0\}$ by setting

$$\nu_i(A) = \overline{\mu}(\pi_0(A))$$

where π_0 is the projection of F onto its base F_0: $\pi_0(x,t) = x$. Recalling that $x \in G_0$ if and only if $x \in F_0$ or $x = T_t(y)$ with $(y,t) \in G(i)$ for some i, we define a $\sigma-$finite Borel measure $\widetilde{\nu}$ on G_0 by taking the sum of $\overline{\mu}$ and the ν_i:

$$\widetilde{\nu}(A) = \overline{\mu}(A \bigcap F_0) + \sum_{i=1}^{\infty} \nu_i(A \bigcap G(i)).$$

Finally we let $\widehat{\nu} = \sigma^*(\widetilde{\nu})$; this is a $\sigma-$finite Borel measure on $G_0 \bigcap U$. Extend $\widehat{\nu}$ to a measure on G_0 by setting $\widehat{\nu}(A) = 0$ for any Borel $A \subseteq G_0 - U$.

Choose a Borel $A \subseteq G_0 \bigcap U$ and let $R = \{(y,t) \in G : y \in A \text{ and } t \in [\alpha, \beta)\}$ be a rectangle in G. The set $R^\star$ of points (y,t) in F such that $\phi(T_t(y)) \in R$ is a union of disjoint sets of the form

$$\{(y,t) \in F : \exists i > 0 \text{ such that } y \in J(i) \text{ and } t \in [j_i(y) + \alpha, j_i(y) + \beta)\}$$

and in particular $R^\star$ satisfies property (1.1) above; therefore

$$(\overline{\mu} \times \lambda)(R^\star) = \mu(\Upsilon(R^\star)) = \mu(\phi^{-1}(R)).$$

But we also have

$$\begin{aligned}
(\overline{\mu} \times \lambda)(R^\star) &= (\beta - \alpha)\overline{\mu}(\pi_0(\sigma^{-1}(A))) \\
&= (\beta - \alpha)\sigma^*(\widetilde{\nu})(A) \\
&= (\beta - \alpha)\widehat{\nu}(A) \\
&= (\widehat{\nu} \times \lambda)(R);
\end{aligned}$$

therefore $\phi^*(\mu)$ and $\widehat{\nu} \times \lambda$ agree on rectangles. As rectangles generate the Borel subsets of G, we have $\phi^*(\mu) = \widehat{\nu} \times \lambda$ and so Theorem 1.6 is established. This argument also proves Theorem 1.7; we have in this case $U = X$, $G_2 = \emptyset$, and $\sigma = id_X$.

2.4. **Continuous representations of Borel semiflows with no IDIs.** Under the hypotheses of Theorem 1.7 we can also give a result modeled after a result of Becker and Kechris [4]. They showed that any Borel action of a Polish *group* on a standard Polish space has a continuous version, that is, that there exists a Polish topology on the phase space with the same Borel structure as the original topology under which the action is jointly continuous. Here, for countable-to-1 Borel semiflows with no IDIs and discrete orbit branchings, we have the same conclusion.

By Theorem 1.7, (X, T_t) is isomorphic to a suspension semiflow (G, S_t) with base $G_0 \subseteq X$, return-time function $g(x)$ and return-time transformation $\widehat{S}$. First, choose a topology $\mathcal{R}_1$ on X which is Polish, stronger than the original topology $\mathcal{T}$, has the same Borel sets as $\mathcal{T}$, and for which G_0 is open. Hence G_0 is a standard Polish space under the relative topology.

Next, place a topology $\mathcal{R}_2$ on G_0 which is Polish, stronger than the relative $\mathcal{R}_1$-topology, has the same Borel sets as the relative $\mathcal{R}_1$-topology, and for which the map $\widehat{S}$ is continuous. Place an even stronger topology $\mathcal{R}_3$ on G_0 which is Polish, stronger than $\mathcal{R}_2$, has the same Borel structure as $\mathcal{R}_2$, and for which g is continuous (see Theorem 13.11 of [10]).

Let $\overline{G} = \{(x,t) \in G_0 \times \mathbb{R}^+ : t \leq g(x)\}$ (endowed with the product of the $\mathcal{R}_3$-topology on G_0 and the usual topology on $\mathbb{R}^+$) and define $\theta : \overline{G} \to G$ by

$$\theta(x,t) = \begin{cases} (x,t) & \text{if } t < g(x) \\ (\widehat{S}(x), 0) & \text{if } t = g(x) \end{cases}$$

and let $\mathcal{R}$ be the largest topology on G which makes θ continuous. Under the $\mathcal{R}$ topology, it is clear that $S_t : G \times \mathbb{R}^+ \to G$ is a jointly continuous semiflow. In fact $\mathcal{R}$ is Polish (in [18], Wagh shows that this topology is Polish when T_t (respectively S_t) is a flow; his proof that the topology so defined is Polish carries through for semiflows).

We have shown the following:

Theorem 2.10. *Let $(X, \mathcal{B}(X), \mu, T_t)$ be a countable-to-1 Borel semiflow with no IDIs that has the discrete orbit branching property. Then there is a Polish topology $\mathcal{R}'$ on X such that the Borel sets of $\mathcal{R}'$ are precisely $\mathcal{B}(X)$ and the semiflow T_t is jointly continuous with respect to $\mathcal{R}'$.*

Proof. We have a Borel isomorphism $\pi : (G, S_t) \to (X, T_t)$ from Theorem 1.7; choose $\mathcal{R}'$ to be the topology which makes π a homeomorphism between $(G, \mathcal{R})$ and $(X, \mathcal{R}')$. □

2.5. **The proof of Theorem 1.8.** In this subsection we now make the additional assumption of Theorem 1.8, namely that there exists a constant $c_0 > 0$ such that orbit branchings along the orbit of a point are always at least time c_0 apart.

Lemma 2.11. *If T_t is a Borel semiflow satisfying the hypotheses of Theorem 1.8, then there exists a section $F_0^\sharp$ for T_t and a number $\delta > 0$ with the following properties:*

1. *The return-time function $r_\sharp(x) : X \to \mathbb{R}^+$ defined by $r_\sharp(x) = \inf\{t > 0 : T_t(x) \in F_0^\sharp\}$ satisfies $r_\sharp(x) \geq \delta$ for all $x \in F_0^\sharp$,*
2. *for any $x \in F_0^\sharp$, $B(x) \bigcap [0, r_\sharp(x)) \subseteq (\delta, r_\sharp(x) - \delta)$, and*
3. *given any $x \in F_0^\sharp$, and $t_1 \neq t_2$ in $B(x) \bigcap [0, r_\sharp(x))$, we have $|t_1 - t_2| > \delta$.*

Proof. Start with any horizontally identifying section F_0 for the semiflow; we denote the return-time function for this section by r_F and the return-time map from F_0 to itself by $\widehat{T}$. Let b and B be such that $r_F(x) \in (b, B)$ for every $x \in F_0$. Now let $\delta = \min(b/10, c_0/10)$.

Now let

$$G'_1 = \{x \in F_0 : \exists i > 0 \text{ such that } j_i(x) \in (r_F(x) - \delta, r_F(x))\}.$$

This is a Borel subset of F_0 since all the j_i are Borel and r_F is Borel. Now let $G' = T_{r_F(x)}(x) : x \in G'_1$. Next define G'' to be the set of all $x \in F_0$ such that $j_i(x) < \delta$ for some $i > 0$. The section is defined as follows:

$$F_0^\sharp = (F_0 - G' - G'') \bigcup T_\delta(G') \bigcup T_{-\delta}(G'')$$

Proof of (1)*:* Suppose $x \in F_0^\sharp$ is such that $r_\sharp(x) = \epsilon < \delta$. Write $x = T_t(y)$ where $y \in F_0$ and $0 \le t < r_F(y)$. We have one of the following:

- *Case 1:* $x \in F_0 - G' - G''$. In this case, $x = y$ and we have $z = T_\epsilon(x) = T_\epsilon(y) \in F_0^\sharp$.
  - *Case 1 (a):* $z \in F_0$. This cannot happen as $\epsilon \le \delta < b < r_F(y)$.
  - *Case 1 (b):* $z \in T_\delta(G') \subseteq T_\delta(F_0)$. This implies $z = T_\delta(w)$ where $w \in F_0$; as $\delta < b < r_F(w)$ and since F_0 is horizontally identifying we have $T_{-\delta}T_\delta(w) \subseteq F_0$; there must then be a point $w' \in T_{-\delta}T_\delta(w)$ with $T_{\delta-\epsilon}(w') = x \in F_0$. This is impossible as $\delta - \epsilon < b < r_F(w')$.
  - *Case 1 (c):* $z \in T_{-\delta}(G'')$. In this case $T_\delta(z) = T_{\delta+\epsilon}(x) \in G'' \subseteq F_0$. But $r_F(x) > b > 2\delta \ge \delta + \epsilon$ so this too is impossible.
- *Case 2:* $x \in T_\delta(G')$. Here, we see that $t = \delta$; we can assume further that $y \in G'$. Let $z = T_\epsilon(x) = T_{\delta+\epsilon}(y)$; $z \in F_0^\sharp$.
  - *Case 2 (a):* $z \in F_0$. This cannot happen as $\epsilon + \delta \le 2\delta < b < r_F(y)$.
  - *Case 2 (b):* $z \in T_\delta(G')$. Here $z = T_\delta(y')$ for some $y \in G' \subseteq F_0$ with $0 \le \delta < r_F(y')$. But we have $z = T_{\delta+\epsilon}(y)$. Since F_0 is horizontally identifying, this implies $\epsilon = 0$ which is impossible (if occurrences of $F_0^\sharp$ accumulated along an orbit, then so would occurrences of F_0).
  - *Case 2 (c):* If $z \in T_{-\delta}(G'')$, then $T_\delta(z) \in F_0$ and therefore $T_{2\delta+\epsilon}(y) \in F_0$. This is impossible since $r_F(y) > b > 3\delta \ge 2\delta + \epsilon$.
- *Case 3:* $x \in T_{-\delta}(G'')$. Write $x' = T_\delta(x) \in F_0$. Let $z = T_\epsilon(x) \in F_0^\sharp$.
  - *Case 3 (a):* $z \in F_0 - G' - G''$. This is impossible, for we would have $T_{\delta-\epsilon}(z) \in F_0$ but also the contradictory fact that $r_F(z) > b > \delta - \epsilon$.
  - *Case 3 (b):* $z \in T_\delta(G')$. Here, we write $z = T_\delta(y)$ for $y \in F_0$; since F_0 is horizontally identifying we see that $T_{-\delta}T_\delta(y) \subseteq F_0$; in particular $T_{\delta-\epsilon}(x) \subseteq F_0$. But then for any $x'' \in T_{\delta-\epsilon}(x)$ we have $x' = T_{2\delta-\epsilon}(x'')$; this is not possible as $r_F(x'') > b > 2\delta - \epsilon$.
  - *Case 3 (c):* $z \in T_{-\delta}(G'')$. Here we have $T_\delta(z) = T_{\delta-\epsilon}(x') \in G'' \subseteq F_0$ which is impossible since $r_F(x') > b > \delta - \epsilon$.

Proof of (2)*:* Let $x \in F_0^\sharp$ and first suppose that $t \in B(x) \bigcap [0, \delta)$.

- *Case 1:* $x \in F_0 - G' - G''$. This is impossible by definition of G''.
- *Case 2:* $x \in T_\delta(G')$. In this case, take $y \in T_{-2\delta}(x)$ and observe that y has an orbit branching at some time ϵ less than δ (this follows from definition of G'). Since $t \in B(x)$, we have also that $t + 2\delta \in B(y)$. By hypothesis the orbit branchings of y must be separated by c_0, but $t + 2\delta - \epsilon < 3\delta < c_0$.

- *Case 3:* $x \in T_{-\delta}(G'')$. Here there is some $\epsilon < \delta$ such that $\epsilon \in B(T_\delta(x))$. So $\epsilon + \delta \in B(x)$ so we have a contradiction as $t \in B(x)$ but $\epsilon + \delta - t < c_0$.

Statement (3) of the Lemma follows immediately from the hypotheses of Theorem 3 since for any x, elements of $B(x)$ are at least c_0 apart, hence at least δ apart. □

Now let $g_\sharp(x) = \min\{r_\sharp(x), j_1(x), j_2(x), j_3(x), ...\}$. $g_\sharp : X \to \mathbb{R}^+$ is Borel. Next let $G_\sharp = \{T_{g_\sharp(x)}x : x \in X\}$; lemma 2.11 ensures that for any $x \in G_\sharp$, we have $g_\sharp(x) \geq \delta$.

Now construct the suspension semiflow with IDI where $G_1 := G_\sharp \bigcap J$ and $G_2 := G_\sharp - J$, $g_\sharp|_{G_1}$ is the return-time function, σ is as in Section 1 and $\widehat{S}(x) = T_{g_\sharp(x)}(x)$. In particular for any $x \in G_1$ we have

$$B(x) \subseteq \{0\} \bigcup \left\{ \sum_{i=0}^{n-1} g_\sharp(\sigma(\widehat{S} \circ \sigma)^i(x)) : n = 1, 2, 3, ... \right\}.$$

Let $\pi : G \to X$ be the map defined by $\pi(x,t) = T_t(x)$. As $G_\sharp$, $g_\sharp$, σ and J are Borel, π is also seen to be a Borel mapping.

Lemma 2.12. *π is a bijection.*

Proof. First we show that π is surjective. Let $x \in X$; write $x = T_t(y)$ where $y \in F_0$ and $0 \leq t < f(y)$ (again, the y is not unique but the t is).

Now for each choice of such a y the set $B(y) \bigcap [0,t]$ is finite. Suppose this set is empty. There are two cases here:

- *Case 1:* $\widehat{S}(y) \in G''$ and $t \geq r_F(y) - \delta$. Here we see that $g_\sharp(y) = r_F(y) - \delta$. If $g_\sharp(y) = t$, then $x \in G_\sharp$, and therefore $(x, 0) \in G$ and $\pi(x, 0) = x$. If $g_\sharp(y) < t$, then $(\sigma(\widehat{S}(y)), t - g_\sharp(y)) \in G$ and $\pi(\sigma(\widehat{S}(y)), t - g_\sharp(y)) = T_{t-g_\sharp(y)}(\sigma(T_{g_\sharp}(y))) = T_t(y) = x$.
- *Case 2:* $\widehat{S}(y) \notin G''$ or $t < r_F(y) - \delta$. In this case $t < g_\sharp(y)$ so $(y,t) \in G$ and $\pi(y,t) = x$.

If $B(y) \bigcap [0,t]$ is nonempty, list the elements of this discrete set in increasing order as $b_{y,1}, b_{y,2}, ..., b_{y,\kappa(y)}$. Notice that $b_{y,\kappa(y)}$ is always the same no matter the choice of y. (If $\kappa(y) = 1$, let $b_{y,0} = 0$.)

- *Case 1:* $b_{y,\kappa(y)} = t$. In this case, choose a particular $y \in T_{-t}(x) \bigcap F_0$; for $\tau > b_{y,\kappa(y)-1}$ we see
$$\min\{j_i(T_\tau(y)) : i > 0\} = t - \tau.$$
So $x \in G_\sharp$ unless $A = T_{[0,t-\tau)}(T_\tau(y))$ intersects $F_0^\sharp$ in at least one point. But A cannot meet F_0 since $r_F(y) > t$ and cannot meet $T_\delta(G')$ or $T_{-\delta}(G'')$ because then y would have orbit branchings too close together, contradicting the hypothesis of this section. So $x \in G_\sharp$; thus $(x, 0) \in G$ and $\pi(x, 0) = x$ as desired.
- *Case 2:* $b_{y,\kappa(y)} < t$. Here we consider the set
$$I^* = \{z = T_{b_{y,\kappa(y)}}(y) : y \in F_0, T_t(y) = x\};$$
for any $z_1, z_2 \in I^*$ we have $(z_1, z_2) \in \textbf{IDI}$. So exactly one point in I^* is in J; call it z_J. We know
$$B(z_J) \bigcap (0, t - b_{y,\kappa(y)}) = \emptyset$$
so $(z_J, t - b_{y,\kappa(y)}) \in G$ (and $\pi(z_J, t - b_{y,\kappa(y)}) = x$) unless $A = T_{(0,t-b_{y,\kappa(y)})}(z_J)$ meets $F_0^\sharp$ in at least one point. But A cannot intersect F_0 nontrivially since

$r_F(y) > t$. If A meets $T_\delta(G')$, then either $t > \delta$ (in which case we have $T_{t-\delta}(y) \in F_0$, a contradiction) or $t \le \delta$ (in which case $b_{y,\kappa(y)} < \delta$, contradicting Lemma 2.11 part (2). If A meets $T_{-\delta}(G'')$ in a point $w = T_\gamma(z_J)$, then as $T_{\gamma+\delta+b_{y,\kappa(y)}}(y) \in F_0$ we have $t-\gamma-b_{y,\kappa(y)} \in [0,\delta)$ whence $(w, t-\gamma-b_{y,\kappa(y)}) \in G$ and $\pi(w, t-\gamma-b_{y,\kappa(y)}) = x$ so in any event π is surjective as desired.

Now we show π is injective. Suppose (x_1,t_1) and (x_2,t_2) are two elements of G with $\pi(x_1,t_1) = \pi(x_2,t_2) = x$. If $t_1 = 0$, then $x \in G_\sharp$ so $T_{t_2}(x_2) \in G_\sharp$. This of course means that $x_2 \in G_\sharp$ and $t_2 = 0$. Since $\pi|_{G_\sharp}$ is the identity, we have $x_1 = x_2$.

On the other hand, if t_1 and t_2 are positive then we suppose without loss of generality that $0 < t_1 \le t_2$. Let $s = \sup\{t \in [0,t_1) : T_{t_1-t}(z_1) = T_{t_2-t}(z_2)\}$; if $s < t_1$ then z_1 has an orbit branching at time $t_1 - s$ which is impossible since $B(z_1) \bigcap (0, g_\sharp(z_1)) = \emptyset$. So $s = t_1$.

Claim: $s = t_2$. If not, consider the point $T_{t_2-s}(z_2)$ which cannot be in $G_\sharp$ since $g_\sharp(z_2) > t_2$. In particular $T_{t_2-s}(z_2) \ne z_1$. But by definition of s, we see that $(T_{t_2-s}(z_2), z_1) \in$ **IDI** so they both have an orbit branching at time 0, hence they are both in $G_\sharp$. This contradicts the above and proves the claim.

Now $s = t_1 = t_2$; if $z_1 \ne z_2$ we see that z_1 has an orbit branching at some time in the interval $[0,t_1]$ which is impossible as $B(z_1) \bigcap (0, g_\sharp(z_1)) = \emptyset$. Thus $(z_1,t_1) = (z_2,t_2)$ and π is injective, hence a bijection as desired. □

To complete the proof of Theorem 1.8 we follow the work of Ambrose [2]. We now have an isomorphism $\phi = \pi^{-1}$ between $(X, \mathcal{B}(X), \mu, T_t)$ and $(G, \mathcal{B}(G), \phi^*(\mu), S_t)$. It remains to show that $\phi^*(\mu)$ is the product of a finite measure on the base with Lebesgue measure in the vertical direction. To accomplish this, we begin with some definitions:

Definition 2.13. Given a set $A \subseteq G_0 \bigcap J$, define A^* to be the set of all (x,t) in G such that $x \in A$. We call A^* the *tube* over A.

Observe that A is a Borel subset of X if and only if A^* is a Borel subset of G.

Definition 2.14. Given two real numbers α and β with $0 < \alpha < \beta$, define $R(\alpha,\beta)$ to be the set of all (x,t) in G with $t \in [\alpha,\beta)$. We call the Borel set $R(\alpha,\beta)$ the $\alpha,\beta-$*strip*.

Choose $c < \delta/2$ and define a measure $\widehat{\nu}$ on the Borel subsets of G_0 by first setting

$$\widehat{\nu}(A) = \frac{1}{c}(\phi^*(\mu))\left(A^* \bigcap R(0,c)\right)$$

for any $A \subseteq G_0 \bigcap J$ (so long as c is small enough, the measure thus defined does not depend on c) and then extending $\widehat{\nu}$ to a measure on G_0 by setting $\widehat{\nu}(A) = 0$ for any Borel $A \subseteq G_0 - J$. This is of course a finite measure.

Proposition 2.15. *$\phi^*(\mu) = \widehat{\nu} \times \lambda$ where λ is Lebesgue measure on the Borel subsets of $\mathbb{R}^+$.*

Proof. Let $A \subset G_0 \bigcap J$ be a Borel set and consider a rectangle $R = A^* \bigcap R(a,b)$. Take a positive integer n; we have

$$R = \bigcup_{i=0}^{n-1} A^* \bigcap R(a + \frac{b-a}{n}, a + 2\frac{b-a}{n}).$$

The sets in this union are disjoint and mapped to one another by the semiflow in a $1-1$ fashion so each has $\phi^*(\mu)-$measure $(1/n)(\phi^*(\mu))(R)$. It follows that for any real number $y \in (a,b)$ we have

$$\phi^*(\mu)(A^* \bigcap S(a,y)) = \frac{y-a}{b-a}\phi^*(\mu)(R). \tag{2}$$

But also for the fixed c we chose above,

$$\begin{aligned}(\widehat{\nu} \times \lambda)(A^* \bigcap R(0,c)) &= \widehat{\nu}(A)\lambda([0,c)) &(3)\\ &= (\phi^*(\mu))(A^* \bigcap R(0,c)). &(4)\end{aligned}$$

Together (2.2)−(2.4) imply that $\widehat{\nu} \times \lambda$ and $\phi^*(\mu)$ agree on any rectangle. Also for any Borel subset of $(G_0 - J) \times \{0\}$ we have $\widehat{\nu} \times \lambda$ and $\phi^*(\mu)$ both equal to zero. Since rectangles and Borel subsets of $(G_0 - J) \times \{0\}$ generate the Borel structure on G, the proposition holds (and the proof of Theorem 1.8 is complete). □

3. **Two counterexamples.**

3.1. **Infinite measures on the base.** We show in this subsection that the measure on the section cannot in general be taken to be finite for an arbitrary Borel semiflow. In particular we examine a specific Borel semiflow based on a transformation φ studied by Boole [5]. Let $\varphi(x) : \mathbb{R} - \{0\} \to \mathbb{R}$ defined by $\varphi(x) = x - \frac{1}{x}$. This transformation preserves Lebesgue measure and is ergodic with respect to Lebesgue measure [1]. Let

$$G_1 = \mathbb{R} - \bigcup_{i=0}^{\infty} \varphi^{-i}(0);$$

G_1 is a G_δ subset of $\mathbb{R}$ hence is Polish under the relative topology. Notice that whenever $x \in G_1$, x has precisely two preimages under φ. For $x \in G_1$ let $f(x) = 2^{-\lfloor |x| \rfloor - 2}$ (here $\lfloor z \rfloor$ means the greatest integer less than or equal to z); f is measurable and satisfies

$$\int_{G_1} f(x)\, dx = 1.$$

Now let S_t be the suspension semiflow (without IDIs) with base G_1, return-time function f, and return-time map φ. This is a Borel semiflow which preserves the product of Lebesgue measures in the base and the vertical direction (in fact it is ergodic with respect to this measure).

Proposition 3.1. *S_t has the discrete orbit branching property.*

Proof. Suppose not; then there exists a point $x \in G$ such that $B(x,0)$, the set of S_t−orbit branchings of $(x,0)$, has an accumulation point. But

$$B(x) = \left\{ \sum_{i=0}^{n-1} f(\varphi^i(x)) : n \in \mathbb{N} \right\}.$$

As f is uniformly bounded away from zero on any bounded subset of G_1, the existence of an accumulation point in $B(x)$ implies $\varphi^i(x) \to \pm\infty$ as $i \to \infty$. But this is impossible, because whenever $|x| > 1$ we have $|\varphi(x)| < |x|$. □

Notice also that the inverse image of any point under any S_t is countable, hence σ−compact, so S_t satisfies the hypotheses of Theorem 1.7.

Theorem 3.2. *S_t is not isomorphic to a suspension semiflow where the base is endowed with a finite Borel measure.*

Proof. Suppose S_t^* is a suspension semiflow with phase space G^* (endowed with a finite Borel measure $\widehat{\mu}^*$), base G_1^*, return-time function f^*, and return-time map $\widehat{S}^*$, isomorphic to S_t. Let $\psi : G \to G^*$ be the desired Borel conjugacy.

It must be the case that $\psi(G_1) \subseteq G_1^*$: if $x \in G_1$, then $S_{-t}(x)$ contains at least two points for all $t > 0$. Therefore $S_{-t}^*\psi(x)$ must contain at least two points for all $t > 0$. But the only such points in G^* are those in G_1^*.

Every $x^* \in \psi(G_1)$ has its first return to $\psi(G_1)$ at some time 2^{-n} for some integer $n \geq 2$ (since the same holds for every $x \in G_1$). Now consider the measurable partition of $\psi(G_1)$ into the sets

$$P_n = \{x \in \psi(G_1) : T_{2^{-n-1}}(x) \in \psi(G_1) \text{ and } T_{(0,2^{-n-1})}(x) \bigcap \psi(G_1) = \emptyset\}$$

for $n = 1, 2, 3, \ldots$. For any n, choose $\alpha \in (0, 2^{-n-1})$. We have

$$\begin{aligned}
2\alpha &= \mu\left(S_{[0,\alpha)}\left(G_1 \bigcap \left((-n-1,-n] \bigcup [n,n+1)\right)\right)\right) \\
&= (\psi^*\mu)\left(\psi\left(S_{[0,\alpha)}\left(G_1 \bigcap \left((-n-1,-n] \bigcup [n,n+1)\right)\right)\right)\right) \\
&= \mu^*(S_{[0,\alpha)}^*(P_n)) \\
&= \alpha\widehat{\mu}^*(P_n);
\end{aligned}$$

whence $\widehat{\mu}^*(P_n) = 2$ for all n. But there are infinitely many disjoint P_n all contained in G_1^*, so $\widehat{\mu}^*$ cannot be a finite measure. □

3.2. **Non-existence of finite-valued step codings.** We end the paper by showing that the independent results of Rudolph [17] and Krengel [12] guaranteeing the existence of a two-valued step coding for measure-preserving flows do not hold for Borel semiflows even if the semiflows under consideration are assumed to be suspension semiflows. In fact, there is no finite-valued step coding in general.

Let $G_1 = [0, 1)$, let $\widehat{\mu}$ be Lebesgue measure on G_1 and let $g : G_1 \to \mathbb{R}^+$ be defined by $g(x) = x + \frac{1}{2}$. Let (G, S_t) be the suspension semiflow (with no IDIs) defined over the base G_1 with return-time function g and return-time transformation $\widehat{S}(x) = 2x$ mod 1. Notice that every $x \in G_1$ has two preimages under $\widehat{S}$ so the orbit branchings of any point z with respect to S_t are the positive times t where $S_t(z) \in G_1$.

Proposition 3.3. *The semiflow S_t defined above does not have a finite-valued step coding; that is, there does not exist a section F for the action for which*

1. *a suspension semiflow with base F is isomorphic to (G, S_t), and*
2. *given any point $z \in F$, the function $r_F(z) = \inf\{t > 0 : S_t(z) \in F\}$ takes on only a finite number of values.*

Proof. Suppose not, and let $a_1, a_2, \ldots, a_n$ be the list of positive real numbers which are the values of $r_F(z)$ for $z \in F$. Let ϕ be the isomorphism from (G, S_t) to the suspension semiflow over F. Observe that F must contain the image under ϕ of all the orbit branchings of S_t. Hence $F \supseteq \phi(G_1)$; since $\phi(G_1)$ is also a section for S_t, for any point $z \in \phi(G_1)$ we therefore have nonnegative integers $m_1, \ldots, m_n$ so that

$$r_{\phi(G_1)}(z) = \sum_{i=1}^{n} m_i a_i$$

where $r_{\phi(G_1)}(z) = \inf\{t > 0 : S_t(z) \in \phi(G_1)\}$. This means that for every $\alpha \in [\frac{1}{2}, \frac{3}{2})$, there are nonnegative integers $m_1(\alpha), ..., m_n(\alpha)$ so that

$$\sum_{i=1}^{n} m_i(\alpha) a_i = \alpha.$$

This is of course impossible as the set of possible $\sum_{i=1}^{n} m_i a_i$ is countable. $\square$

Acknowledgements. The author extends his thanks to the referees for helpful comments concerning a previous draft of this paper.

REFERENCES

[1] R. Adler and B. Weiss. *The ergodic infinite measure preserving transformation of Boole*, Israel J. Math. **16** (1973), 263-278.
[2] W. Ambrose. *Representation of ergodic flows*, Ann. of Math. II **42** (1941), 723-729.
[3] W. Ambrose and S. Kakutani. *Structure and continuity of measurable flows*, Duke Math. J. **9** (1942), 25-42.
[4] H. Becker and A. Kechris. "The Descriptive Set Theory of Polish Group Actions", London Math. Society Lecture Note Series. Cambridge University Press, 1996.
5. G. Boole. *On the comparison of transcendents with certain applications to the theory of definite integrals*, Philos. Trans. Roy. Soc. London **147** Part III (1857), 745-803.
6. L. Christiano and S. Harrison. *Chaos, sunspots and automatic stabilizers*, J. of Monetary Economics **44** (August 1999), 3-31.
[7] J. Feldman. *New $K-$automorphisms and a problem of Kakutani*, Israel J. Math. **24** (1976), 16-38.
[8] S. Kakutani. *Induced measure preserving transformations*, Proc. Imp. Acad. Tokyo **19** (1943), 635-641.
[9] J. Kammeyer and D. Rudolph. "Restricted orbit equivalence for actions of discrete amenable groups", Cambridge University Press, 2002.
[10] A. Kechris. "Classical Descriptive Set Theory", Graduate Texts in Mathematics. Springer, 1995.
[11] U. Krengel. *Darstellungssätze für Strömungen und Halbströmungen. II*, Math. Ann. **182** (1969), 1-39.
[12] U. Krengel. *On Rudolph's representation of aperiodic flows*, Ann. Inst. H. Poincaré Sect. B. (N.S.) **12** (1976).
[13] C.-H. Lin and D. Rudolph. *Sections for semiflows and Kakutani shift equivalence*, in "Modern Dynamical Systems and Applications", Cambridge University Press, 2004.
14. B. Miller and C. Rosendal. *Descriptive Kakutani equivalence*, to appear, J. Eur. Math. Soc.
[15] M. Nadkarni. "Basic ergodic theory", Birkhäuser Verlag, 1978.
[16] D. Ornstein, D. Rudolph and B. Weiss. *Equivalence of measure-preserving transformations*, Mem. Amer. Math. Soc., vol. 262 (1982).
[17] D. Rudolph. *A two-valued step coding for ergodic flows*, Math. Z. **150** (1976), 201-220.
[18] V. Wagh. *A descriptive version of Ambrose's representation theorem for flows*, Proc. Indian Acad. Sci. Math. Sci. **98** (1988), 101-108.

Received xxxx 20xx; revised xxxx 20xx.
E-mail address: dmm@math.northwestern.edu