ON THE MAXIMAL WEYL COMPLEXITY OF FAMILIES OF FOUR POLYNOMIALS

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ABSTRACT. We show that any family of four essentially distinct integer polynomials has Weyl complexity no greater than 4.

A generalization of Szemerédi's [11] theorem on arithmetic progressions of Bergelson and Leibman [1] states that given any family $\mathcal{P} = \{p_1, ..., p_k\}$ of polynomials with integer coefficients satisfying $p_j(0) = 0$ for all j, and given any subset A of $\mathbb{Z}$ of positive upper Banach density[1], A must contain infinitely many sets of the form

$$\{x, x + p_1(n), ..., x + p_k(n)\}.$$

This result is closely associated with the behavior of multiple ergodic averages of the form

$$\frac{1}{N-M} \sum_{n=M}^{N-1} T^{p_1(n)} f_1 \cdot ... \cdot T^{p_k(n)} f_k. \tag{0.1}$$

Bergelson and Leibman studied these averages in enough detail to establish their result on polynomial progressions, but their work left some interesting unanswered questions (notably, whether or not the limit of (0.1) exists as $N - M \to \infty$, and what the limit is if it exists).

An important idea in the study of multiple ergodic averages such as (0.1), dating to Furstenberg's proof [4] of Szemerédi's [11] theorem, is that of a characteristic factor:

Definition. *Let $(X, \mathcal{X}, \mu, T)$ be a dynamical system[2]. A* characteristic factor *for the averages (0.1) is a $T-$invariant sub $\sigma-$algebra $\mathcal{Y}$ of $\mathcal{X}$ such that the the averages (0.1) converge to 0 in $L^2(\mu)$ as $N - M \to \infty$ whenever $E(f_j|\mathcal{Y}) = 0$ for some j.*

Roughly speaking, this means the L^2-limit of (0.1) is the same when the functions f_j are replaced by their conditional expectation on the factor $\mathcal{Y}$. Ideally then, by explicitly describing some class of systems which serve as characteristic factors for the averages under consideration, and by examining the averages for those systems, one could obtain results about the averages for general systems. In fact, this has been done in [6] (for weak convergence) and [9] (for L^2-convergence):

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[1] The *upper Banach density* of a set $A \subset \mathbb{Z}$ is $d^*(A) = \lim_{N\to\infty} \sup_{M\in\mathbb{Z}} \frac{1}{N}|A \cap [M, M + N)|$.

[2] A *dynamical system* is for us a Lebesgue probability space $(X, \mathcal{X}, \mu)$ with an invertible transformation $T : X \to X$ which is measurable ($A \in \mathcal{X} \Rightarrow T^{-1}A \in \mathcal{X}$) and measure-preserving ($A \in \mathcal{X} \Rightarrow \mu(A) = \mu(T^{-1}A)$). Systems may be denoted simply (X, T) or even just X or T when the other elements of the system are clear.

Theorem (Host-Kra [6], Leibman [9])**.** *Let $\mathcal{P} = \{p_1, ..., p_k\}$ be a family of essentially distinct polynomials. Then there is a $d = d(\mathcal{P})$ such that for every ergodic system $(X, \mathcal{X}, \mu, T)$, there is a characteristic factor for the averages (0.1) which is an inverse limit of $d-$step nilsystems.*

These results are motivated by work in [5] which describes characteristic factors in the case where $\mathcal{P} = \{n, 2n, ..., kn\}$, and are useful because combined with another result of Leibman [8] one obtains the convergence of the averages (0.1) in $L^2(\mu)$. However, an actual computation of the limit of these averages is difficult because the evaluation of this limit in an arbitrary nilsystem is nontrivial. One problem related to this issue is that the work in [6] and [9] does not provide useful upper bounds on the minimum value of d which works in the above theorem for any large class of polynomial families, i.e. does not characterize the "smallest" characteristic factor which works for the averages.

A useful notion to attack this question is the Weyl complexity of a family of polynomials, introduced in [2]. Roughly speaking, the Weyl complexity of a family of polynomials codes the smallest m for which the "$(m-1)^{th}$ Host-Kra factor" Z_{m-1} (see [5] or [7]) is characteristic for the polynomial ergodic averages in the situation where the dynamical system under consideration is a Weyl system. Exactly how the Host-Kra factors are constructed is not central to our discussion; what is important is that Z_{m-1} is the inverse limit of $(m-1)-$step nilsystems. It is conjectured, but not known, that the Weyl complexity also determines the smallest m for which Z_{m-1} is characteristic for arbitrary systems; the best result known in this vein is a result of Leibman ([10], section 13) which guarantees that if the Weyl complexity of a family of polynomials is m, then an $l-$step nilsystem is characteristic for the averages where $l \leq ((m-1)^3 + (m-1)^2)/2$.

In [2] Bergelson, Leibman and Lesigne conjectured that the Weyl complexity of a family of k essentially distinct polynomials ("essential distinctness" means that each p_j is nonconstant and that no two p_j differ by a constant) must be at most k. This is clear for $k = 2$; in [3] Frantzikinakis showed that families of three essentially distinct polynomials have Weyl complexity at most three; in this paper we show:

Theorem 1. *Let $\mathcal{P}$ be any family of four essentially distinct polynomials. Then $\mathcal{P}$ has Weyl complexity at most* 4.

As a corollary, using the result of Leibman [10] described above, we obtain the fact that there is a characteristic factor for the averages (0.1) which is an inverse limit of 18-step nilsystems.

1. Weyl complexity

In this section we define the Weyl complexity of a family of essentially distinct polynomials, and state some results demonstrating how this notion connects the machinery of characteristic factors to the structure of certain orbits in Weyl dynamical systems.

A *connected Weyl system* is the action of an ergodic, nilpotent, affine transformation on a finite-dimensional torus (preserving Lebesgue measure). A *standard Weyl system of level d* is an transformation $T : \mathbb{T}^d \to \mathbb{T}^d$ defined by $T(x_1, ..., x_d) = (x_1 + \alpha, x_2 + x_1, x_3 + x_2, ..., x_d + x_{d-1})$ where $\alpha \in \mathbb{T}$ is irrational. A *quasi-standard Weyl system of level d* is a transformation $T : \mathbb{T}^d \to \mathbb{T}^d$ defined by

$T(x_1,...,x_d) = (x_1+\alpha_1, x_2+m_{2,1}x_1+\alpha_2, ..., x_d+\sum_{j=1}^{d-1} m_{d,j}x_j+\alpha_d)$ where $\alpha_j \in \mathbb{T}$, $\alpha_1 \notin \mathbb{Q}$, and $m_{j,j-1} \neq 0$ for all $j > 1$.

Given a family P of m polynomials (henceforth, a *polynomial* means any polynomial with integer coefficients) and a system (X, T), we define the *orbit of a polynomial family with respect to* T as the subset of X^{m+1} defined by

$$\mathcal{O}(P,T) = \{(x, T^{p_1(n)}x, ..., T^{p_m(n)}x) : x \in X, n \in \mathbb{N}\}.$$

Definition. *Let $P = \{p_1, ..., p_m\}$ be a family of essentially distinct polynomials. The Weyl complexity of this family is the number $W(P) \in \mathbb{N}$ with the following properties:*

(1) *$W(P)$ is the minimal $r \in \mathbb{N}$ such that for every $d \geq r$, there is a quasi-standard Weyl system (X, T) of level d such that*

$$\{(0, ..., 0, x_r, ..., x_d)\}^{k+1} \subseteq \overline{\mathcal{O}(P^0, T)}$$

where $P^0 = \{p_1 - p_1(0), ..., p_m - p_m(0)\}$.

(2) *$W(P)$ is the minimal $r \in \mathbb{N}$ such that for every quasi-standard Weyl system (X, T) of any level $d \geq r$ we have*

$$\{(0, ..., 0, x_r, ..., x_d)\}^{k+1} \subseteq \overline{\mathcal{O}(P^0, T)}$$

where $P^0 = \{p_1 - p_1(0), ..., p_m - p_m(0)\}$.

(3) *$W(P)$ is the maximal $s \in \mathbb{N}$ (or 1 if there is no s) such that for some quasi-standard Weyl system of level $s-1$, there exist characters χ_i of $X = \mathbb{T}^d$, at least one of which depends nontrivially on the coordinate x_{s-1}, such that*

$$\chi_0(x)\chi_1(T^{p_1(n)}x) \cdot ... \cdot \chi_m(T^{p_1(n)}x) = 1$$

for all $x \in \mathbb{T}^{s-1}$.

(4) *$W(P)$ is the minimal $m \in \mathbb{N}$ such that for every connected Weyl system (X, T), the factor Z_{m-1} is characteristic for L^2-convergence or weak convergence of the corresponding polynomial ergodic averages.*

The coincidence of the first two definitions is shown in [2]; that the third is equivalent to the first two is straightforward (in particular, the word "some" in definition (3) could be replaced with the word "every"); the equivalence of the fourth definition with the first two is also in [2].

The following lemma is an easy consequence of the fourth description of Weyl complexity and is proven in [3]:

Lemma 1.1. *If $\mathcal{P} = \{p_1, ..., p_m\}$ is a family of essentially distinct polynomials, then $W(P) = W(p_1 - p_m, p_2 - p_m, ..., p_{m-1} - p_m, -p_m)$.*

The fourth description of Weyl complexity also makes it apparent that the Weyl complexity of a polynomial family does not depend on the order in which the polynomials are written.

2. Notation and general setup

Let $\mathcal{P}$ be a family of k essentially distinct polynomials. Let d_j be the degree of p_j for $j = 1, ..., k$; we assume the polynomials are ordered so that $d_1 \geq d_2 \geq ... \geq d_m$. For each j, write $p_j(x) = a_{j,0}x^{d_j} + a_{j,1}x^{d_j-1} + ... + a_{j,d_j-1}x + a_{j,d_j}$. In particular, $a_{j,0}$ represents the leading coefficient of p_j.

Let $\alpha \in [0,1)$ be irrational and let $T : \mathbb{T}^k \to \mathbb{T}^k$ be the quasi-standard Weyl system defined by

$$[T(x_1,...,x_k)]_j = x_j + \sum_{i=1}^{j-1} \binom{j}{i} x_i + \alpha.$$

We see that

$$[T^r(x_1,...,x_k)]_j = x_j + \sum_{i=1}^{j-1} \binom{j}{i} r^{j-i} x_i + r^j \alpha. \tag{2.1}$$

For $z \in \mathbb{T}^k$ let $e(z) = e^{2\pi i z}$ and define characters χ_j of $\mathbb{T}^m$ by

$$\chi_j(x_1,...,x_k) = e(c_{i1}x_1 + ... + c_{ik}x_k);$$

suppose further that

$$\chi_0(x) \prod_{j=1}^{k} \chi_j(T^{p_j(n)}x) = 1. \tag{2.2}$$

Substituting (2.1) into (2.2) and collecting like terms we obtain the following system of k equations:

$$\begin{aligned}
\sum_{j=1}^{k} k c_{jk} p_j &= 0 \\
\sum_{j=1}^{k} \binom{k-1}{k-2} c_{j,k-1} p_j + \sum_{j=1}^{k} \binom{k}{k-2} c_{jk} p_j^2 &= 0 \\
\vdots \quad & \quad \vdots \\
\sum_{j=1}^{k} \binom{k-l+1}{k-l} c_{j,k-l} p_j + ... + \sum_{j=1}^{k} \binom{k}{k-l} c_{jk} p_j^l &= 0 \\
\vdots \quad & \quad \vdots \\
\sum_{j=1}^{k} \binom{3}{2} c_{j3} p_j + \sum_{j=1}^{k} \binom{4}{2} c_{j4} p_j^2 + ... + \sum_{j=1}^{k} \binom{k}{2} c_{jk} p_j^{k-2} &= 0 \\
\sum_{j=1}^{k} 2c_{j2} p_j + \sum_{j=1}^{k} 3c_{j3} p_j^2 + ... + \sum_{j=1}^{k} k c_{jk} p_j^{k-1} &= 0 \\
\sum_{j=1}^{k} c_{j1} p_j + \sum_{j=1}^{k} c_{j2} p_j^2 + ... + \sum_{j=1}^{k} c_{jk} p_j^k &= 0
\end{aligned}$$

We will refer to these equations respectively as the $l = 1$ equation, $l = 2$ equation, etc. (so that the last equation is the $l = k$ equation). Collectively we call this system the *fundamental system*. To show that the Weyl complexity of $\mathcal{P}$ is at most k, we need only show that the fundamental system has no solution (for the c_{ij}) over the integers for which at least one $c_{ik} \neq 0$. This we do in the next section, for the case $k = 4$.

3. Proof for four polynomials

The fundamental system when $k = 4$ is:

$$0 = c_{14}p_1 + c_{24}p_2 + c_{34}p_3 + c_{44}p_4 \tag{3.1}$$

$$0 = \sum_{j=1}^{4} c_{j3}p_j + 2\sum_{j=1}^{4} c_{j4}p_j^2 \tag{3.2}$$

$$0 = 2\sum_{j=1}^{4} c_{j2}p_j + 3\sum_{j=1}^{4} c_{j3}p_j^2 + 4\sum_{j=1}^{4} c_{j4}p_j^3 \tag{3.3}$$

$$0 = \sum_{j=1}^{4} c_{j1}p_j + \sum_{j=1}^{4} c_{j2}p_j^2 + \sum_{j=1}^{4} c_{j3}p_j^3 + \sum_{j=1}^{4} c_{j4}p_j^4 \tag{3.4}$$

We divide the result into cases depending on the degrees of the polynomials in the family, tackling each case separately.

Case 1: $d_1 > d_2 > d_3 \geq d_4$. From equation (3.1) we see that $c_{14} = c_{24} = 0$ and therefore that $d_3 = d_4$. Plugging this information into equation (3.2) we obtain

$$0 = \sum_{j=1}^{4} c_{j3}p_j + 2(c_{34}p_3^2 + c_{44}p_4^2).$$

If $c_{13} \neq 0$, then we see that $d_1 = 2d_3$. But then the right hand side of equation (3.3) has degree $4d_3$ which is impossible. Therefore $c_{13} = 0$. If $c_{23} \neq 0$, we see similarly that $d_2 = 2d_3$. From looking at equation (3.3) we see that $d_1 = 2d_2 = 4d_3$ and therefore equation (3.4) has degree $8d_3$ which is impossible. So $c_{13} = c_{23} = 0$. Then by considering the role of the leading coefficients of the polynomials p_3 and p_4 in equations (3.1) and (3.2), we obtain $c_{34} = -c_{44}$ and therefore $p_3 = p_4$, contradicting the essential distinctness of the polynomials.

Case 2: $d_1 > d_2 = d_3 > d_4$. Again, we have $c_{14} = 0$. Suppose $c_{13} \neq 0$. Then from equation (3.3) we have $3d_2 = 2d_1$ but from equation (3.4) we have $4d_2 = 3d_1$; these statements contradict one another so in fact $c_{13} = 0$.

Equations (3.1) and (3.2) then imply

$$\begin{cases} c_{24}a_{2,0} + c_{34}a_{3,0} = 0 \\ c_{24}a_{2,0}^2 + c_{34}a_{3,0}^2 = 0 \end{cases};$$

therefore $a_{2,0} = a_{3,0}$ and $c_{24} = -c_{34}$. Let $d_0 = \max\{r : a_j = b_j \, \forall j \leq r\}$; the preceding argument ensures $d_0 \geq 1$. Suppose $d_0 < d$, then we can write $p_2 = q + p_2'$ and $p_3 = q + p_3'$ where $q(x) = a_{2,0}x^d + \ldots + a_{d_0}x^{d-d_0+1}$. Let d_2' and d_3' be the degrees of p_2' and p_3', respectively. Assume $d_2' \geq d_3'$ (otherwise reorder the polynomials) so that we have $d_2' > 0$. Substitute all this into equation (3.1) to obtain $c_{24}(p_2' - p_3') = 0$. This is impossible since p_2' and p_3' are either of different degrees or have different leading coefficients. Therefore $d_0 = d$ and $p_2 = p_3$, a contradiction.

Case 3: $d_1 > d_2 = d_3 = d_4$. Again, $c_{14} = 0$. By the same argument as in the first paragraph of Case 2, $c_{13} = 0$.

Suppose that $d_1 \geq 3d_2$. Then there is a term in equation (3.4) with coefficient c_{12} of degree at least $6d_2$ which cannot be cancelled with any other term; in this case we have $c_{12} = 0$. Therefore, either $d_1 < 3d_2$ or $c_{12} = 0$. Let a_j be the leading coefficient of p_j for $j = 2, 3, 4$. The first three equations of the fundamental system lead to

$$c_{24}a_2^i + c_{34}a_3^i + c_{44}a_4^i = 0$$

for $i = 1, 2, 3$ (the case where $i = 3$ is a consequence of the fact that either $d_1 < 3d_2$ or $c_{12} = 0$). There must therefore be two numbers from a_2, a_3 and a_4 which coincide; by reordering the polynomials if necessary we have $a_2 = a_3$. Now by Proposition 1.1 we see that

$$W(P) = W(p_1 - p_2, -p_2, p_3 - p_2, p_4 - p_2).$$

This new collection of polynomials must satisfy the conditions of either Case 1 or Case 2.

Case 4: $d_1 = d_2 > d_3 \geq d_4$. By looking at the leading coefficients of equations (3.1) and (3.2), we see that $c_{14}a_{1,0} + c_{24}a_{2,0} = 0$ and $c_{14}a_{1,0}^2 + c_{24}a_{2,0}^2 = 0$. This implies that $c_{14} = -c_{24}$ and that $a_{1,0} = a_{2,0}$. Write $p_1 = q + p_1'$ and $p_2 = q + q_2'$ as was done in Case 2. Making these substitutions in equation (3.2), we see that the left hand side of (3.2) has a single $4c_{14}q(p_1' - p_2')$ which is of degree $d_1 + d_1'$.

If $c_{14} \neq 0$, the this term must be cancelled with another term of the same degree which must be coming from some p_t^2 for $t = 3$ or 4 (in particular t is the smallest number > 2 for which $c_{t4} \neq 0$). Hence

$$2d_t = d_1 + d_1'.$$

However, in equation (3.3), the highest power coming from p_1^3 and p_2^3 has degree $2d_1 + d_1'$ and the highest power term coming from the other terms is $3d_t$. So we have also

$$3d_t = 2d_1 + d_1';$$

this equation together with $2d_t = d_1 + d_1'$ implies $d_1 = d_t$ which is impossible. Consequently $c_{14} = 0$ and also $c_{24} = 0$. Knowing this, equation (3.1) ensures that $d_3 = d_4$ (otherwise $c_{34} = c_{44} = 0$).

Suppose that $d_1 < 2d_3$. Then by considering the role of leading coefficients in equations (3.1) and (3.2) we see that $a_{3,0}c_{34}^i + a_{4,0}c_{44}^i = 0$ for $i = 1, 2$. This implies that $c_{34} = -c_{44}$ and then equation (3.1) gives $p_3 = p_4$, contradicting the essential distinctness of the polynomials. Therefore $d_1 \geq 2d_3$.

Consequently the highest degree term in (3.4) must come from $c_{13}p_1^3$ and $c_{23}p_2^3$; we expand the sum of these two terms to obtain

$$\begin{aligned} c_{13}p_1^3 + c_{23}p_2^3 &= c_{13}(q + p_1')^3 + c_{23}(q + p_2')^3 \\ &= (c_{13} + c_{23})q^3 + \text{ terms of degree } < 3d_1. \end{aligned}$$

From this we can conclude that $c_{13} = -c_{23}$ and that the highest degree of the preceding expression is $2d_1 + d_1'$ (coming from the term $3c_{13}q^2(p_1' - p_2')$). Equation (3.1) then reduces to

$$c_{13}(p_1' - p_2') + 2c_{34}p_3^2 + 2c_{44}p_4^2 = 0.$$

We next show $d_1' = 2d_3$.

Suppose $d_1' > 2d_3$. Then from the reduced version of (3.1) we have $c_{13} = 0$ and then by looking at the leading coefficients of equations (3.1) and (3.2) we again have $c_{34} = -c_{44}$ and $p_3 = p_4$, a contradiction.

Suppose $d_1' < 2d_3$. Then by looking at the leading coefficients of equations (3.1) and (3.2) we again have $c_{34} = -c_{44}$ and $p_3 = p_4$, the same contradiction. Hence $d_1' = 2d_3$.

Knowing this, we see the highest power term of (3.4) (coming from $c_{13}p_1^3 + c_{23}p_2^3$) has degree $2d_1 + d_1' > 4d_3$. The coefficient of this term is a multiple of c_{13}, so $c_{13} = 0$. But again, by looking at the leading coefficients of equations (3.1) and (3.2) we again have $c_{34} = -c_{44}$ and $p_3 = p_4$, a contradiction.

Case 5: $d_1 = d_2 = d_3 > d_4$. From analyzing the leading coefficients of equations (3.1), (3.2) and (3.3) we see

$$c_{14}a_{1,0}^i + c_{24}a_{2,0}^i + c_{34}a_{3,0}^i = 0$$

for $i = 1, 2, 3$. This can only occur if at least two of the numbers $a_{1,0}, a_{2,0}$ and $a_{3,0}$ are equal; without loss of generality we can assume $a_{1,0} = a_{2,0}$. Now looking at the leading coefficients of equations (3.1) and (3.2) we have

$$\begin{cases} (c_{14} + c_{24})a_{1,0} + c_{34}a_{3,0} = 0 \\ (c_{14} + c_{24})a_{1,0}^2 + c_{34}a_{3,0}^2 = 0 \end{cases}$$

This leads to one of two possible situations:

Case 5 (a): $a_{1,0} = a_{3,0}$. In this case, we apply Proposition 1.1 to get

$$W(\mathcal{P}) = W(-p_1, p_2 - p_1, p_3 - p_1, p_4 - p_1).$$

The first and last polynomials of this new family have degree d_1, but the middle two polynomials have degree less than d_1, so Case 4 applies to this new family.

Case 5 (b): $a_{1,0} \neq a_{3,0}$. This implies $c_{14} = -c_{24}$ and $c_{34} = 0$. Write $p_1 = q + p_1'$ and $p_2 = q + p_2'$ as has been done in previous cases. Now consider equation (3.2) which reduces to

$$\sum_{j=1}^{4} c_{j3}p_j + 2[c_{14}q(p_1' - p_2') + c_{14}(p_1'^2 - p_2'^2) + c_{44}p_4^2] = 0.$$

This means either $c_{14} = 0$ (impossible since this would imply $c_{j4} = 0$ for all j) or $d_1 + d_1' = 2d_4$.

Consider also the fourth sum in equation (3.4) which becomes

$$c_{14}[(p_1' + q)^4 - (p_2' + q)^4] + c_{44}p_4^4;$$

after expanding and combining terms we obtain

$$4c_{14}q^3(p_1' - p_2') + \text{ terms of degree} < 3d_1 + d_1' + c_{44}p_4^4$$

and can conclude that $3d_1 + d_1' = 4d_4$.

But the two equations $d_1 + d_1' = 2d_4$ and $3d_1 + d_1' = 4d_4$ taken together imply that $d_1 = d_1'$, a contradiction.

Case 6: $d_1 = d_2 = d_3 = d_4$. From analyzing the leading coefficients of all four equations (3.1)-(3.4) we see

$$c_{14}a_{1,0}^i + c_{24}a_{2,0}^i + c_{34}a_{3,0}^i + c_{44}a_{4,0}^i = 0$$

for $i = 1, 2, 3, 4$. This can only occur if at least two of the polynomials p_j share the same leading coefficient. After reordering the polynomials we can assume $a_{1,0} = a_{2,0}$. Now by Proposition 1.1 we have

$$W(\mathcal{P}) = W(-p_1, p_2 - p_1, p_3 - p_1, p_4 - p_1).$$

The second polynomial in this new family has smaller degree than the first polynomial so one of the previous cases applies. This exhausts all possible cases so the result is proved.

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