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CONTINUITY OF CONDITIONAL MEASURES ASSOCIATED TO MEASURE-PRESERVING SEMIFLOWS

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Abstract. Let X be a standard probability space and T_t a measure-preserving semiflow on X. We show that there exists a set X_0 of full measure in X such that for any $x \in X_0$ and $t \geq 0$ there are measures $\mu^+_{x,t}$ and $\mu^-_{x,t}$ which for all but a countable number of t give a distribution on the set of points y such that $T_t(y) = T_t(x)$. These measures arise by taking weak$^*-$limits of suitable conditional expectations. Say that a point x has a *measurable orbit discontinuity* at time t_0 if either $\mu^+_{x,t}$ or $\mu^-_{x,t}$ is weak$^*-$ discontinuous in t at t_0. We show that there exists an invariant set of full measure in X such that any point in this set has at most countably many measurable orbit discontinuities. Furthermore we show that if x has a measurable orbit discontinuity at time 0, then x has an orbit discontinuity at time 0 in the sense of [1] .

1. Introduction

Let $(X, \mathcal{F}, \mu)$ be a standard Lebesgue space and $T_t : X \times \mathbb{R}^+ \to X$ be a $\mathcal{F}-$measurable, $\mu-$preserving semiflow on X. The following two questions motivate this paper:

(1) Given a point $x \in X$ and some time $t \geq 0$, does there exist a natural "distribution" (measure) on the set of points identified with x at time t (i.e. the set of all $y \in X$ such that $T_t(y) = T_t(x)$)?
(2) Given a fixed $x \in X$, how do these distributions change as t increases? In particular, are the measures weak$^*-$continuous in t?

We approach the questions posed above by using weak$^*-$limits of conditional expectations to define for each point x (in a set of full measure) two "measure paths" $\mu^+_{x,t}$ and $\mu^-_{x,t}$ which give a natural distribution on the set of points which are identified with x at all times greater than t, and the set of points identified with x at some time less than t, respectively. (We note that we do not specifically answer question (1) above in that we do not explicitly construct measures on the set of points identified with x *at time* t.) We say x has a "measurable orbit discontinuity" at time t_0 if the two measures μ^+_{x,t_0} and μ^-_{x,t_0} differ; these are precisely the times at which $\mu^+_{x,t}$ or $\mu^-_{x,t}$ is weak$^*-$discontinuous in t. Our main result is:

Theorem 1. *Given a measure-preserving semiflow $(X, \mathcal{F}, \mu, T_t)$ on a Lebesgue space, there exists an invariant set X' of full measure in X such that for every $x \in X'$, x has at most countably many measurable orbit discontinuities.*

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We consider an example which describes what is meant by "measurable orbit discontinuity". Start with the circle $S^1 = I/\partial I$ and let $\widehat{\mathcal{F}}$ be the Lebesgue $\sigma-$algebra of S^1. Let $X = (S^1 \times [0,1])/\sim$ where $(z,1) \sim (2z \bmod 1, 0)$ and let T_t be the suspension semiflow on X over the map $\widehat{T} : S^1 \to S^1$ taking z to $2z \bmod 1$ with constant height function (equal to 1). This action is measurable and preserves the product of Lebesgue measure on S^1 with Lebesgue measure on $[0,1]$. Now given a point $(z,s) \in X$ (assume $s \neq 0,1$), we see that the only point identified with (z,s) at times $0 \leq t < 1-s$ is (z,s) itself. However, if $t \in [1-s, 2-s)$, there are two points identified with (z,s) at time t. The measure we seek in the first question above should therefore be, in this case, atomic, and supported on the two points (z,s) and $((z+1/2) \bmod 1, s)$ with masses $1/2$ and $1/2$. This makes sense, as the conditional expectation $E(f|\widehat{T}^{-1}(\widehat{\mathcal{F}}))(z)$ is given by $\frac{1}{2}f(z) + \frac{1}{2}f((z+\frac{1}{2}) \bmod 1)$. Similarly, for any integer $n > 0$, for $t \in [n-s, n+1-s)$ the measure we seek should be atomic, supported on 2^n points each having mass 2^{-n}. Hence at each time $n-s$ where n is a positive integer we see that these measures are weak$^*-$discontinuous in t. So the point (z,s) is thought of as having a countably infinite number of "measurable orbit discontinuities". In fact every point in this example has infinitely many measurable orbit discontinuities, so Theorem 1 is in this sense the strongest statement that can be made about the prevalence of measurable orbit discontinuities along the orbits of an arbitrary measure-preserving semiflow.

Questions (1) and (2) above are motivated by a broader program to study the structure of general measure-preserving semiflows. In particular, understanding the structure and prevalence of discontinuous behavior is relevant to the problems of universally modeling semiflows (see [1], [2]) and building an isomorphism theory for such actions. (Any measurable conjugacy between two semiflows must preserve measurable orbit discontinuities.) In [1] similar ideas of "continuous" and "discontinous" behavior in semiflows were explored from the viewpoint of topology. Given a Borel measurable semiflow T_t on a standard Polish space X, it was shown that for any $x \in X$ the set of points identified with x at time t by the semiflow grows continuously (in some sense) in t except for at most a countable number of times t. The times where $T_{-t}T_t(x)$ grows discontinuously are called the *orbit discontinuities* of x. In Section 3 of this paper, we show that if a point has a measurable orbit discontinuity at time 0, then it must also have an orbit discontinuity there in the sense of [1] as well.

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2. Measurable orbit discontinuities

First denote for each $t > 0$ the $\sigma-$algebras

$$\mathcal{F}_t = T_{-t}(\mathcal{F}) = \{A \subseteq X : A = T_{-t}(B) \text{ for some } B \in \mathcal{F}\}.$$

A set A belongs to $\mathcal{F}_t$ if and only if $T_{-t}T_t(A) = A$. In particular observe that $\mathcal{F}_0$ is the original $\sigma-$algebra $\mathcal{F}$ and that as t increases, the $\mathcal{F}_t$ get smaller. Extend this notation to negative t by setting $\mathcal{F}_t = \mathcal{F}_0$ for $t < 0$.

Now consider the Rohklin decompositions of $(\mathcal{F}, \mu)$ over the subalgebras $\mathcal{F}_t$. By this we mean that for each t, X can be conjugated measurably to the unit square $I^2 = [0,1] \times [0,1]$ where the $\mathcal{F}_t-$measurable sets correspond to subsets of I^2 of the form $A \times [0,1]$ where A is Lebesgue measurable in I [3]. The measure μ can then be written as

$$\mu = \int_0^1 \mu_{x,t}\, dx$$

where $\mu_{x,t}$ is the corresponding fiber measure for the point x with respect to $\mathcal{F}_t$ and dx is a Lebesgue measure on $[0,1]$ (by "a" Lebesgue measure we mean a measure which is isomorphic to the usual Lebesgue measure on $[0,1]$ with a possible addition of up to countably many atoms). The measures $\mu_{x,t}$ (if they exist) are defined by

$$\int f\, d\mu_{x,t} = E(f|\mathcal{F}_t)(x).$$

Any $\mu-$integrable function $f : X \to \mathbb{R}$ is therefore $\mu_{x,t}-$measurable for $\mu-$almost every x by Fubini's theorem and satisfies

$$\int f\, d\mu \; = \int_0^1 \int f\, d\mu_{x,t}\, dx = \int_0^1 E(f|\mathcal{F}_t)(x)\, dx.$$

In particular $\mu_{x,t} = \delta_x$ (a point mass at x) whenever $t \leq 0$.

Of course there is a problem concerning the *existence* of the fiber measures $\mu_{x,t}$. Conditional expectations of the form $E(f|\mathcal{F}_t)$ are only guaranteed to exist for almost every x; consequently it is only immediate that given $t \geq 0$, for $\mu-$almost every x, the measure $\mu_{x,t}$ is defined as above. We would like to "reverse the quantifiers" and characterize a set of full measure in X on which $\mu_{x,t}$ can be defined for every t. First for each $t \geq 0$ define

$$G_t = \{x \in X : E(f|\mathcal{F}_t)(x) \text{ is uniquely defined for any } \mu\text{−integrable } f\}.$$

We think of G_t as a set of x for which $\mu_{x,t}$ "exists". Since $\mu(G_t) = 1\,\forall\, t$, we have $\mu(\bigcap_{q\in\mathbb{Q}^+} G_q) = 1$.

Lemma 2.1. *Let $x \in X$ and $t \geq 0$ be such that $x \in G_t$. Then for $0 \leq s \leq t$, $T_s(x) \in G_{t-s}$ and $\mu_{T_s(x),t-s} = T_s(\mu_{x,t})$.*

Proof. Let f be an integrable function. Then

$$\begin{aligned}\int f\, d(T_s(\mu_{x,t})) &= \int (f \circ T_s)\; d\mu_{x,t} \\ &= E(f \circ T_s|\mathcal{F}_t)(x) \\ &= E(f|\mathcal{F}_{t-s})(T_s(x)) \\ &= \int f\, d\mu_{T_s(x),t-s}\end{aligned}$$

as desired. □

As a consequence we see that $T_s(\mu_{x,t})$ is a point mass for $s \geq t$.

Corollary 2.2. *$X_0 = \{x \in X : x \in G_t$ for a dense set of $t \in \mathbb{R}\}$ is forward invariant under T_t.*

Proof. It suffices to show that $A = X - X_0$ is backward invariant. Let $x \in A$; there exists an interval $S \subseteq \mathbb{R}^+$ such that $x \notin G_s$ for all $s \in S$. Let $y \in T_{-t}(x)$. Given $s \in S, \mu_{y,t+s}$ cannot exist; otherwise $x \in G_s$ exists by Lemma 2.1. Thus for any time in the set $S+t$ (which is of positive Lebesgue measure) y has no fiber measure, so $y \in A$. □

Now X_0 contains $(\bigcap_{q\in\mathbb{Q}^+} G_q)$, so $\mu(X_0) = 1$. Therefore X_0 is an invariant set of full measure, so we can without loss of generality assume $X_0 = X$. So for any x, the fiber measure $\mu_{x,t}$ exists for a dense set $G(x) \supseteq \mathbb{Q}^+$ of t in $[0,\infty)$. Now we describe how to "fill in" the gaps where the measure is not guaranteed to exist. Notice that for each $t \in G(x)$, $E(f|\mathcal{F}_t)$ exists for any measurable f. Now define for each nonnegative integer k the sets

$$\mathbb{G}_k = \left\{ \frac{j}{2^k} : j \in \mathbb{Z}, j \geq 0 \right\}$$

and let $\mathbb{G} = \bigcup_{k=1}^{\infty} \mathbb{G}_k$. Notice the following:

(1) $\mathbb{G}_k \subseteq \mathbb{G}_{k+1}$ for all k.
(2) For all k, $\mathbb{G}_k \subseteq G(x)$ for every $x \in X$.
(3) $\overline{\mathbb{G}} = \mathbb{R}^+$.

For any $f : X \to [0,1]$ that is $\mathcal{F}$-measurable, we can define a function $E_x(f) : \mathbb{G} \to [0,1]$ by

$$E_x(f)(d) = E(f|\mathcal{F}_d)(x) = \int f \, d\mu_{x,t}.$$

We need an analog of Doob's Upcrossing lemma (our version deals with downcrossings) to establish the existence of right- and left- hand limits of $E_x(f)$ at any time t. A function h whose domain and range are subsets of $\mathbb{R}$ has n *downcrossings* of the interval $[a,b]$ $(a \neq b)$ if there exist lists of numbers $a_1, ..., a_n$ and $b_1, ..., b_n$ in the domain of h with $a_i < b_i < a_{i+1} \, \forall i$ and $h(a_i) \geq b, h(b_i) < a \, \forall i$. Similarly we say h has n *upcrossings* of the interval $[a,b]$ if there exist lists of numbers $a_1, ..., a_n$ and $b_1, ..., b_n$ in the domain of h with $a_i < b_i < a_{i+1} \, \forall i$ and $h(a_i) \leq b, h(b_i) > a \, \forall i$. If a function has n downcrossings of $[a,b]$, then it must have (at least) $n-1$ upcrossings of that interval, and vice versa.

Proposition 2.3. *Given any* $[a,b] \subseteq [0,1]$,

$$\mu\left(\{x : E_x(f) \text{ has } m \text{ downcrossings of } [a,b]\}\right) \leq \left(\frac{1-b}{1-a}\right)^m.$$

Proof. For each $d \in \mathbb{G}_k$ define

$$A_{k,d} = \{x \in X : E_x(f)(d) \geq b \text{ and } E_x(f)(\delta) < b \text{ for all } \delta < d \text{ in } \mathbb{G}_k\}.$$

$A_{k,d}$ is the set of points for which the function $E_x(f)|_{\mathbb{G}_k}$ first crosses above the interval $[a,b]$ at time d. In particular, it is a stopping time which is $\mathcal{F}_d$-measurable. Now let

$$\overline{A_{k,d}} = \{x \in A_{k,d} : E_x(f)(\delta) < a \text{ for some } \delta > d \text{ in } \mathbb{G}_k\}.$$

The set $\overline{A_{k,d}}$ indicates those points in $A_{k,d}$ which eventually cross beneath the interval $[a,b]$. For any $x \in \overline{A_{k,d}}$, there must be a least δ in $\mathbb{G}_k$ greater than d for which $E_x(f)(\delta) < a$. Call this number $\Delta_{k,d}(x)$. Now

$$\int_{A_{k,d}} f \, d\mu = \int_{A_{k,d} - \overline{A_{k,d}}} f \, d\mu + \int_{\overline{A_{k,d}}} f \, d\mu; \tag{2.1}$$

manipulating the left-hand side of (2.1) we get

$$\int_{A_{k,d}} f \, d\mu = \int_{A_{k,d}} E_x(f)(d) \, d\mu \geq b \cdot \mu(A_{k,d}).$$

As for the right-hand side of (2.1), we note that since $f \leq 1$,

$$\int_{A_{k,d}-\overline{A_{k,d}}} f\, d\mu \leq \mu(A_{k,d}) - \mu(\overline{A_{k,d}}),$$

and by the definition of $\overline{A_{k,d}}$,

$$\int_{\overline{A_{k,d}}} f\, d\mu = \int_{\overline{A_{k,d}}} E_x(f)(d)\, d\mu \leq a \cdot \mu(\overline{A_{k,d}}).$$

Putting this all together, equation (2.1) becomes the inequality

$$b \cdot \mu(A_{k,d}) \leq \mu(A_{k,d}) - (1-a)\mu(\overline{A_{k,d}})$$

which can be rewritten to obtain

$$\frac{\mu(\overline{A_{k,d}})}{\mu(A_{k,d})} \leq \frac{1-b}{1-a}.$$

In particular, this means that only a fraction $(1-b)/(1-a)$ of the points x for which $E_x(f)$ crosses above b at time d can have $E_x(f)$ cross below a after time d. Using this fact, we proceed inductively. Given a finite list $d_1, ..., d_m$ of elements of $\mathbb{G}_k$, we define the sets $A_{k,(d_1,...,d_m)}$ and $\overline{A_{k,(d_1,...,d_m)}}$ and the function $\Delta_{k,(d_1,...,d_m)} : \overline{A_{k,(d_1,...,d_m)}} \to \mathbb{G}_k$ inductively as follows:

$$A_{k,(d_1,...,d_m)} = \begin{cases} A_{k,d_1} & \text{if } m = 1 \\ \{x \in \overline{A_{k,(d_1,...,d_{m-1})}} : E_x(f)(d_m) \geq b \text{ and} & \\ \quad E_x(f)(\delta) < b \text{ for all } \delta \in (\Delta_{k,(d_1,...,d_{m-1})}, d)\} & \text{if } m > 1 \end{cases}$$

$$\overline{A_{k,(d_1,...,d_m)}} = \{x \in A_{k,(d_1,...,d_m)} : E_x(f)(\delta) \leq a \text{ for some } \delta > d_m \text{ in } \mathbb{G}_k\}$$

$$\Delta_{k,(d_1,...,d_m)}(x) = \min\{\delta > d_m : E_x(f)(\delta) \leq a\}$$

The set $A_{k,(d_1,...,d_m)}$ is the set of points x for which $E_x(f)$ first becomes at least b at time d_1, then drops below a (at time $\Delta_{k,d_1}(x)$), then next becomes at least b at time d_2, then drops below a, then next becomes at least b at time d_3, etc. Inside each set $A_{k,(d_1,...,d_m)}$ we pick out those points for which $E_x(f)$ drops below a again after time d_m and call them $\overline{A_{k,(d_1,...,d_m)}}$. For any point in this set, there is a first time where $E_x(f) \leq a$; this time is called $\Delta_{k,(d_1,...,d_m)}(x)$.

Now by the same argument as given above, we see that

$$\mu(\overline{A_{k,(d_1,...,d_m)}}) \leq \left(\frac{1-b}{1-a}\right) \mu(A_{k,(d_1,...,d_m)}).$$

Hence

$$\begin{aligned} \mu\left(\bigcup_{d_m > d_{m-1}} \overline{A_{k,(d_1,...,d_m)}}\right) &\leq \left(\frac{1-b}{1-a}\right) \mu\left(\bigcup_{d_m > d_{m-1}} A_{k,(d_1,...,d_m)}\right) \\ &\leq \left(\frac{1-b}{1-a}\right) \mu(\overline{A_{k,(d_1,...,d_{m-1})}}). \end{aligned}$$

Applying the argument again we see

$$\mu\left(\bigcup_{d_m>d_{m-1}>d_{m-2}} \overline{A_{k,(d_1,\ldots,d_m)}}\right) \leq \left(\frac{1-b}{1-a}\right)\mu\left(\bigcup_{d_{m-1}>d_{m-2}} \overline{A_{k,(d_1,\ldots,d_{m-1})}}\right)$$
$$\leq \left(\frac{1-b}{1-a}\right)^2 \mu(A_{k,(d_1,\ldots,d_{m-2})})$$
$$\leq \left(\frac{1-b}{1-a}\right)^2 \mu(\overline{A_{k,(d_1,\ldots,d_{m-2})}})$$

and inductively

$$\mu\left(\bigcup_{(d_1,\ldots,d_m)\in\mathbb{G}_k} \overline{A_{k,(d_1,\ldots,d_m)}}\right) \leq \left(\frac{1-b}{1-a}\right)^m.$$

Let

$$S_{m,k} = \{x : E_x(f)|_{\mathbb{G}_k} \text{ has } m \text{ downcrossings of } [a,b]\};$$

in fact this set is equal to

$$\bigcup_{(d_1,\ldots,d_m)\in\mathbb{G}_k} \overline{A_{k,(d_1,\ldots,d_m)}}$$

so $\mu(S_{m,k}) \leq \left(\frac{1-b}{1-a}\right)^m$ for all k, m. Finally if we let

$$S_m = \{x : E_x(f) \text{ has } m \text{ downcrossings of } [a,b]\},$$

we observe $S_m = \bigcup_k S_{m,k}$ and $S_{m,k} \subseteq S_{m,k+1}$ so

$$\mu(S_m) = \lim_{k\to\infty} \mu(S_{m,k}) \leq \left(\frac{1-b}{1-a}\right)^m.$$

as desired. □

Corollary 2.4. *Let f be $\mathcal{F}_0-$measurable and $E_x(f)$ defined as above. For $\mu-$almost every $x \in X$, the function $E_x(f)(d)$ has left- and right-hand limits at every $t \in \mathbb{R}^+$, i.e. there exist numbers L^- and L^+ so that:*

- *Given any $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $d \in (t-\delta, t) \bigcap \mathbb{G}$, $|E_x(f)(d) - L^-| < \epsilon$.*
- *Given any $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $d \in (t, t+\delta) \bigcap \mathbb{G}$, $|E_x(f)(d) - L^+| < \epsilon$.*

Proof. Let $l^- = \liminf_{d\to t^-} E_x(f)(d)$ and $l^+ = \limsup_{d\to t^+} E_x(f)(d)$. Define

$$X_0(f) = \{x \in X : \forall \alpha, \beta \in \mathbb{Q}, E_x(f) \text{ has only finitely many downcrossings of } [\alpha, \beta]\}.$$

Let S_m be as in the previous proposition; then

$$X_0(f) = X - \bigcup_{\alpha\in\mathbb{Q}} \bigcup_{\beta\in\mathbb{Q},\beta>\alpha} \bigcap_{m=1}^{\infty} S_m.$$

By the previous proposition $\mu\left(\bigcap_{m=1}^{\infty} S_m\right) = 0$ so $X_0(f)$ has full measure in X. If $l^+ \neq l^-$, we can choose rational numbers α and β with

$$l^- + \frac{l^+ - l^-}{4} < \alpha < \beta < l^+ - \frac{l^+ - l^-}{4}.$$

The function $E_x(f)$ must have infinitely many downcrossings of the interval $[\alpha, \beta]$ so x cannot lie in $X_0(f)$. $\square$

This result allows us to extend (for $\mu-$almost every x) the function $E_x(f) : \mathbb{G} \to [0,1]$ to the reals in two ways by taking limits as in the previous corollary. First, we define the right-continuous function $E_x^+(f) : \mathbb{R} \to [0,1]$ by setting

$$E_x^+(f)(t) = \lim_{d \to t^+} E_x(f)(d).$$

Similarly, define left-continuous $E_x^-(f) : \mathbb{R} \to [0,1]$ by setting

$$E_x^-(f)(t) = \lim_{d \to t^-} E_x(f)(d).$$

Observe that for $x \in X_0(f), \lim_{s \to t^-} E_x^+(f)(s) = E_x^-(f)(t)$. For if not, there exist sequences s_n and t_n of points in $\mathbb{G}$ with $s_n \to t^-$ and $t_n \to t^-$ with $E_x(f)(s_n)$ close to $E_x^-(f)(t)$ but $E_x(f)(t_n)$ uniformly bounded away from $E_x^-(f)(t)$. Consequently $E_x(f)$ has infinitely many downcrossings of some interval; this is impossible since $x \in X_0(f)$. Similarly $\lim_{s \to t^+} E_x^-(f)(t) = E_x^+(f)(t)$ for $x \in X_0(f)$ so $E_x^+(f)(t)$ is continuous at t if and only if $E_x^-(f)(t)$ is continuous at t if and only if $E_x^+(f)(t) = E_x^-(f)(t)$. Furthermore, since both $E_x^+(f)$ and $E_x^-(f)$ have left- and right-hand limits at every t, any discontinuities of either of these functions are necessarily jump discontinuities.

Corollary 2.5. *For any f which is $\mathcal{F}_0-$ measurable, $E_x^+(f)$ and $E_x^-(f)$ have only countably many discontinuities for any $x \in X_0(f)$.*

Proof. Suppose t is a point of discontinuity for $E_x^+(f)$. Then there exist rational numbers α, β in between $\lim_{d \to t^-} E_x(f)(d)$ and $\lim_{d \to t^+} E_x(f)(d)$ such that $E_x(f)$ has either an upcrossing or downcrossing of $[\alpha, \beta]$. However, for $x \in X_0(f)$, every such rational interval can only be crossed by $E_x(f)$ a finite number of times. Since there are only countably many choices for α and β, $E_x(f)$ can only have countably many discontinuities. $\square$

Take a countable family of continuous functions $\mathbf{F} = \{f_i\}_{i=1}^{\infty}$ mapping X into $[0,1]$ whose linear span is dense in $L^1(X, \mu)$. By Corollary 2.5, for each $f_i \in \mathbf{F}$ there is a set X_i of full measure in X such that $E_x(f_i)$ has only countably many discontinuities. Let $X_0 = \bigcap_i X_i$ (this is a set of full measure in X); then for each $x \in X_0$ define

$$C(x) = \{t : E_x(f_i) \text{ is continuous at } t \text{ for every } f_i\};$$

the complement of $C(x)$ is countable.

We now have two mappings from $X \times \mathbf{F} \times \mathbb{R}$ into $[0,1]$ defined by $(x, f, t) \mapsto E_x^+(f)(t)$ and $(x, f, t) \mapsto E_x^-(f)(t)$. Fix x and t; the resulting mappings $f \mapsto E_x^+(f)(t)$ and $f \mapsto E_x^-(f)(t)$ are bounded functionals since $|f_i| \leq 1$ and are linear by the linearity of the conditional expectation operator. Hence by the Riesz representation theorem they extend to probability measures $\mu_{x,t}^+$ and $\mu_{x,t}^-$ on X.

Proposition 2.6. *For every $x \in X_0$ and every t, $\mu_{x,t}^+$ and $\mu_{x,t}^-$ are the left- and right-hand weak$^*-$ limits of the $\mu_{x,t}$. More precisely, fix x and t and let d^* be a metric for the weak$^*-$ topology on X. Then:*

- *For every $\epsilon > 0$, there is a $\delta > 0$ so that for every $s \in (t, t+\delta)$ for which $\mu_{x,s}$ exists, $d^*(\mu_{x,s}, \mu_{x,t}^+) < \epsilon$.*

- *For every $\epsilon > 0$, there is a $\delta > 0$ so that for every $s \in (t, t-\delta)$ for which $\mu_{x,s}$ exists, $d^*(\mu_{x,s}, \mu^-_{x,t}) < \epsilon$.*

Proof. Let $f : X \to [0,1]$ be continuous. Then there exists a sequence f_i with $f_i \to f$ in $L^1(X)$ and each f_i in the linear span of the $\mathbf{F}$. Fix $\epsilon > 0$ and let $s > t$. Then:

$$\begin{aligned}\left|\int f\, d\mu^+_{x,t} - \int f\, d\mu_{x,s}\right| &\leq \left|\int (f - f_i)\, d\mu^+_{x,t}\right| + \left|\int f_i\, d\mu^+_{x,t} - \int f_i\, d\mu_{x,s}\right| + \\ &\qquad \left|\int (f - f_i)\, d\mu_{x,s}\right| \\ &\leq \int |f - f_i|\, d\mu^+_{x,t} + |E^+_x(f_i)(t) - E_x(f_i)(s)| \\ &\qquad + \int |f - f_i|\, d\mu_{x,s}.\end{aligned}$$

The outer expressions in this final expression are less than $\epsilon/3$ if i is chosen large enough, and the interior summand is less than $\epsilon/3$ if s is chosen close enough to t by Corollary 2.4. Thus $\int f\, d\mu_{x,s} \to \int f\, d\mu^+_{x,t}$ as $s \to t^+$ as desired. The proof that $\mu^-_{x,t}$ is the left-hand limit is similar. □

Now for each $x \in X_0$ we have two *measure paths* of x: the measures $\mu^+_{x,t}$ which are weak* right-continuous, and the measures $\mu^-_{x,t}$ which are weak* left-continuous. We say that x has a *measurable orbit discontinuity at time* t_0 if $\mu^+_{x,t_0} \neq \mu^-_{x,t_0}$.

Proposition 2.7. *The following are equivalent:*

(1) *x has no measurable orbit discontinuity at time t_0.*
(2) *The measure path $\mu^+_{x,t}$ is weak*−continuous at t_0.*
(3) *The measure path $\mu^-_{x,t}$ is weak*−continuous at t_0.*

Proof. Notice $\mu^+_{x,t_0} = \mu^-_{x,t_0}$ if and only if the weak*-limits of $\mu_{x,t}$ as t approaches t_0 from both the right and left are the same. The assumption that either $\mu^+_{x,t}$ or $\mu^-_{x,t}$ is continuous at t_0 is equivalent to the equality of the left- and right-hand weak*-limits. □

Notice that for $t \in C(x)$, $\mu^+_{x,t} = \mu_{x,t} = \mu^-_{x,t}$ so therefore x cannot have a measurable orbit discontinuity at time t. Consequently we immediately see the following:

Proposition 2.8. *Every $x \in X_0$ has only countably many measurable orbit discontinuities.*

Proposition 2.9. *Suppose $x \in X$ has a measurable orbit discontinuity at time t_0. Then for any $z \in T_{-s}(x)$, z has a measurable orbit discontinuity at time $s + t_0$.*

Proof. Recall that by Lemma 2.1 we know that $\mu_{x,t} = T_s(\mu_{z,t+s})$ so long as the first measure exists. Consequently by taking weak*−limits as $t \to t_0$ from both the left and right we obtain

$$\mu^+_{x,t_0} = T_s(\mu^+_{z,s+t_0})$$

and

$$\mu^-_{x,t_0} = T_s(\mu^-_{z,s+t_0}).$$

By assumption $\mu^+_{x,t_0} \neq \mu^-_{x,t_0}$. Therefore $\mu^-_{z,s+t_0} \neq \mu^+_{z,s+t_0}$ so z has a measurable orbit discontinuity at time t_0 as desired. □

As a consequence, we see that the set of points x which have only countably many measurable orbit discontinuities is an invariant set. From Proposition 2.8 we know that this set X' is of full measure in X (it contains X_0) so we have established Theorem 1.

3. Orbit discontinuities: measure theory versus topology

We now examine the relationship between orbit discontinuities in the sense of [1] and the measurable orbit discontinuities constructed here. Let X be a standard Polish space and let ν be any probability measure on X such that all the Borel subsets of X are $\nu-$measurable; we define the *support* of ν, denoted *supp*(ν), to be the complement of all open sets in X which have $\nu-$measure zero. Notice that for any open $A \subseteq X$ disjoint from $T_{-t}T_t(x)$, $\mu_{x,t}(A) = E(A|\mathcal{G}_t)(x) = 0$. Consequently the support of $\mu_{x,t}$ is contained in the closure of $T_{-t}T_t(x)$.

Lemma 3.1. *supp*$(\mu^+_{x,t}) \subseteq \overline{\bigcap_{s>t} T_{-s}T_s(x)}$.

Proof. Recall first that the support of each $\mu_{x,t}$ is contained in $\overline{T_{-t}T_t(x)}$. Let t_n be a decreasing sequence of numbers converging to t from above for which μ_{x,t_n} exists for every n; consequently $\mu^+_{x,t}$ is the weak$^*-$limit of the μ_{x,t_n}.

Let $A = X - \overline{\bigcap_{s>t} T_{-s}T_s(x)}$. If $A = \emptyset$ we are done. Otherwise let A' be any nonempty closed set contained in A; by the Urysohn lemma there exists a continuous function f on X such that $f = 0$ on $\overline{\bigcap_{s>t} T_{-s}T_s(x)}$ and $f = 1$ on A'. Notice that $\int f\, d\mu_{x,t_n} = 0$ for every n; therefore $\int f\, d\mu^+_{x,t} = 0$ since $\mu^+_{x,t}$ is the weak$^*-$limit of the μ_{x,t_n}. But also

$$\int f\, d\mu^+_{x,t} \geq \mu^+_{x,t}(A')$$

so $\mu^+_{x,t}(A') = 0$. But since X is a metric space, A can be written as the increasing union of closed sets contained in A. Therefore $\mu^+_{x,t}(A) = 0$. □

We now give a correspondence between measurable orbit discontinuities and topological orbit discontinuities. Of course, measurable orbit discontinuities are defined for actions on Lebesgue spaces and orbit discontinuities are defined for Borel actions on Polish spaces, so we must assume here that the space under consideration has both the structure of a Lebesgue space and standard Polish space.

Proposition 3.2. *Let X be a standard Polish space; let $\mathcal{B}(X)$ be the $\sigma-$algebra of Borel subsets of X and suppose that $\mathcal{F}$ be a $\sigma-$algebra containing $\mathcal{B}(X)$ such that $(X, \mathcal{F}, \mu)$ is a Lebesgue space. Suppose T_t is an action of $\mathbb{R}^+$ on X such that $(X, \mathcal{B}(X), \mu, T_t)$ is a Borel semiflow and $(X, \mathcal{F}, \mu, T_t)$ is a measure-preserving semiflow. If $x \in X$ has a measurable orbit discontinuity at time 0, then x has an orbit discontinuity at time 0.*

Proof. By hypothesis $\mu^+_{x,0} \neq \delta_x$. Consequently $\mu^+_{x,0}$ must be supported on a set strictly larger than $\{x\}$. Let $z \in \text{supp}(\mu^+_{x,0}) - \{x\}$. Then by the preceding lemma $z \in \overline{\bigcap_{t>0} T_{-t}T_t(x)}$ so there exist a sequence of points $z_n \in X$ with $z_n \to z$ and $T_{1/n}(z_n) = T_{1/n}(x)$. Denote by i the inclusion $i_T^{\mathbb{Q}^+} : X \to X_1^{\mathbb{Q}^+}$ and consider the sequence $i(z_n)$ in $X_1^{\mathbb{Q}^+}$; let ζ be the limit of any subsequence $i(z_{n_k})$ which converges; $\zeta(0) = \lim_{k\to\infty} i(z_{n_k})(0)$ since the mapping from $X_1^{\mathbb{Q}^+}$ to X taking f to $f(0)$ is continuous. Therefore $\zeta(0) = z$ so in particular there is no subsequence of the $i(z_n)$ converging to $i(x)$. Consequently there exists a $\delta > 0$ and an $N > 0$ such that for all $n > N$, $d(i(z_n), i(x)) > \delta$.

Take a refining, generating sequence $\mathcal{P}_k$ of partitions for $X^{\mathbb{Q}^+}$. Choose k large enough such that the maximum diameter of a $\mathcal{P}_k-$atom is less than $\delta/4$. For every rational $q > 0$, $\sigma_{-q}\sigma_q(i(x))$ intersects an atom of $\mathcal{P}_k$ which is d_M-distance at least δ from the atom of $\mathcal{P}_k$ containing x, namely an atom containing an $i(z_n)$. Such an atom cannot contain x, so we see x must have an orbit discontinuity at time 0. $\square$

It is unknown if anything more general can be said in this context. If a point x has a measurable orbit discontinuity at time $t_0 > 0$, we can conclude using reasoning along the lines of the proof of Proposition 3.2 that for every $s > t_0$ there is at least one point y_s with $T_s(y_s) = T_s(x)$ but $T_t(y_s) \neq T_t(x)$ for every $t < t_0$. However, it could be the case that the sequence $i_T^{\mathbb{Q}^+}(y_s)$ is the limit of points z_n in $X_1^{\mathbb{Q}^+}$ with $\sigma_{t_n}(z_n) = \sigma_{t_n}(i(x))$ for $t_n < t_0$, in which case x would not have an orbit discontinuity at time t_0.

Consider also this (admittedly simple) example which illustrates that topological orbit discontinuities can occur where there is no measurable orbit discontinuity. Let Ω_L be the set of functions f from $[0,\infty)$ into $\{0,1\}$ for which there exists a number $c = c(f) \in [0,1)$ such that $f(t)$ is constant on every interval of the form $[0,\infty)\bigcap(c+i, c+i+1]$ for $i \in \mathbb{Z}$. (This is the same as the space Ξ_L constructed in Section 3 of [1] without the "marker".) We put a metric on Ω_L by

$$d(f,f') = \int_0^\infty \frac{|f(t) - f'(t)|}{e^t}\,dt;$$

this makes Ω_L a Polish space. The semiflow σ_t is defined on Ω_L by the shift $\sigma_t(f)(s) = f(t+s)$; this is a Borel action. Let δ_1 be the Dirac measure assigning mass 1 the σ_t-fixed point $g(x) \equiv 1$ and 0 to the rest of the space; our Borel measure-preserving semiflow is $(\Xi_R, \delta_1, \sigma_t)$.

The (topological) orbit discontinuities of this action do not depend on the measure; every $f \in \Xi_L$ has infinitely many orbit discontinuities at the times $c, c+1, c+2, \ldots$. But the function $g \equiv 1$ has no measurable orbit discontinuities; for every t we have $\mu^+_{g,t} = \mu^-_{g,t} = \delta_1$. (The set of full measure on which the measure paths exist can be taken to be the fixed point g.)

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