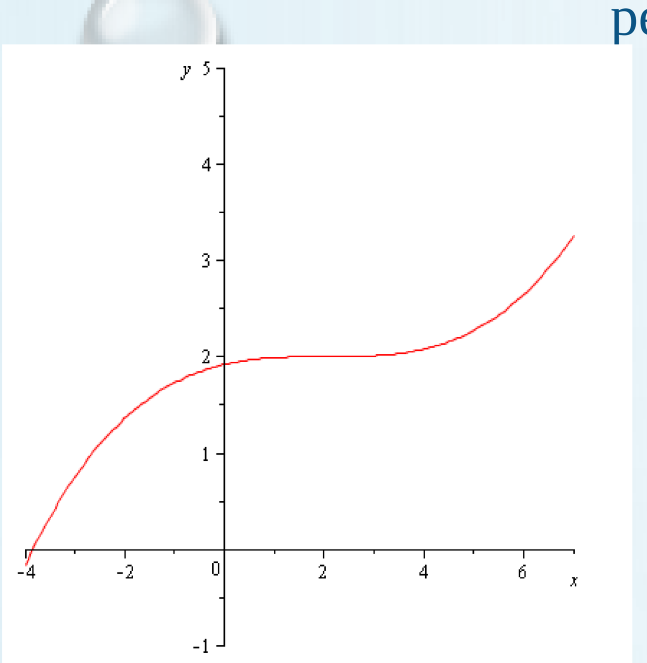
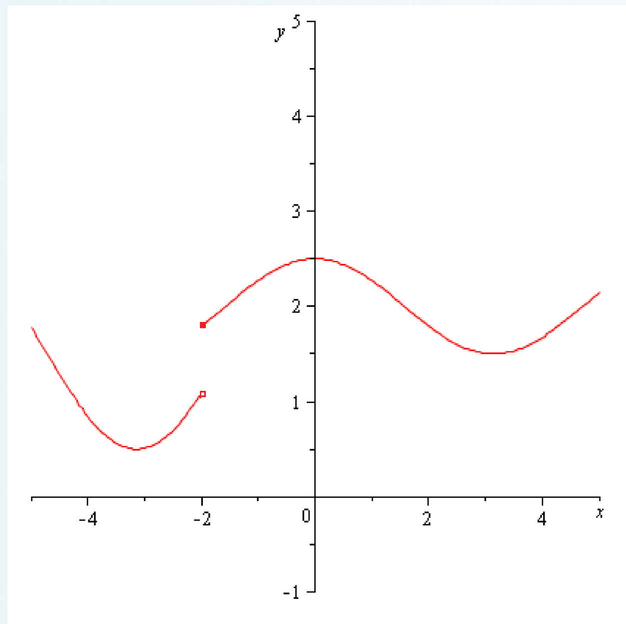
The Raindrop Function: Looks Can Be Deceiving...

Mickelle Bradley & Jaime Mullen
Department of Mathematics
Ferris State University

Continuity

Can the function be drawn without picking up your pencil?



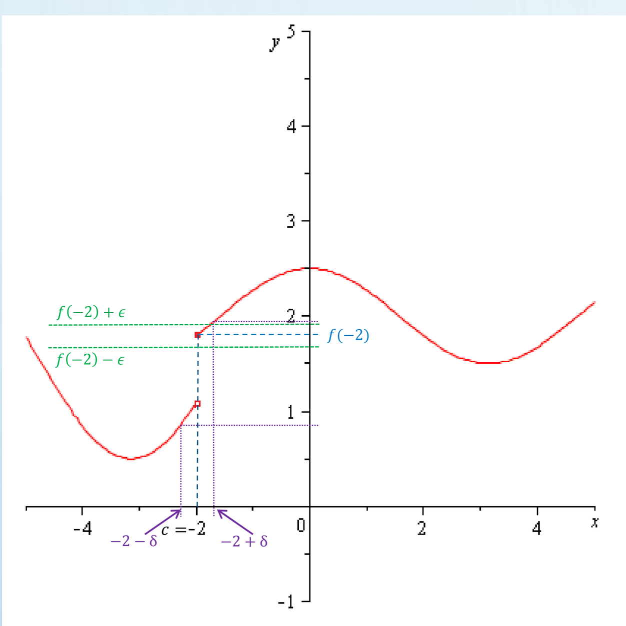


Are these graphs continuous?

Yes No

More Precisely:

Given c, we say that f is continuous at c if for any $\epsilon > 0$ we can find a $\delta > 0$ such that if you move at most δ away from c in any direction, the outputs stay within ϵ of $f(c)$.



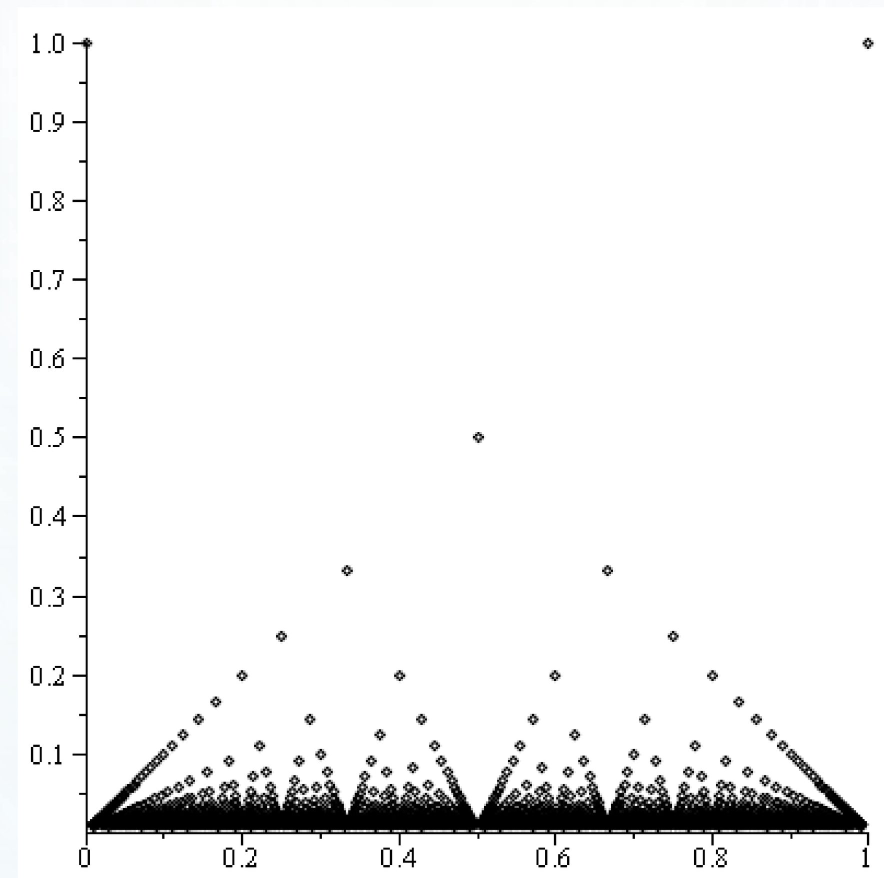
Because no purple region stays entirely within the region between the green lines, we are able to say that the function is not continuous at $x = -2$.

What About the Raindrop Function?

We can easily see that at the rational numbers values of the raindrop function are far apart and do not fulfill the criteria to be called continuous. However, at the irrationals we have only values of zero and we are able to find rational numbers between the irrationals that are close enough to zero in order to be continuous. Therefore, the raindrop function is not continuous at the rational numbers, but continuous at the irrationals.

The Raindrop Function

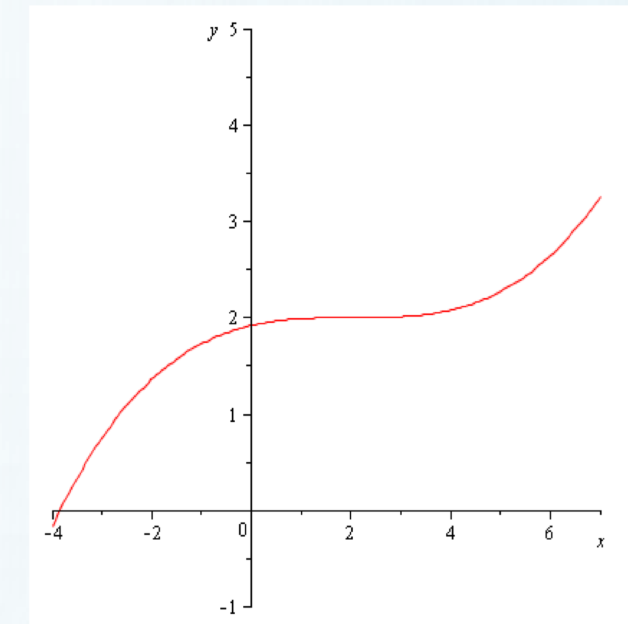
$$f(x) = \begin{cases} \dfrac{1}{q} & \text{if } x = \dfrac{p}{q} \text{ in lowest terms with } p, q \in \mathbb{Z}, q > 0 \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

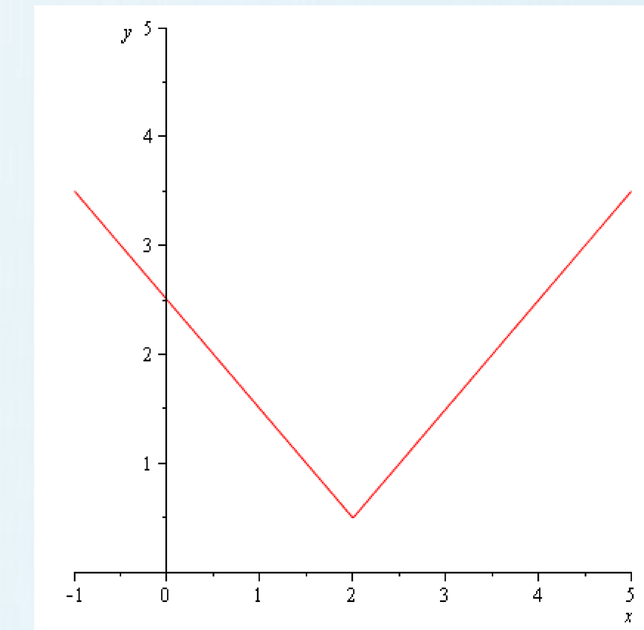


We discuss the distinctions between notions (such as continuity, differentiability, and integrability) described intuitively in calculus, and the rigorous definitions of these ideas coming from higher-level mathematics. As an example, we investigate properties of an unusual function called the raindrop function.

Differentiability

Is the function smooth?





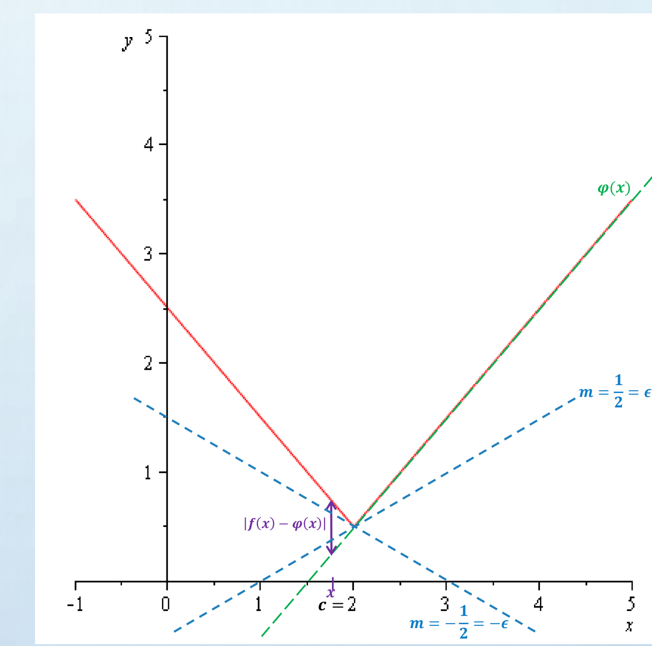
Are these graphs differentiable?

Yes No

More precisely:

Given c, we say that f is differentiable at c if there is a line φ such that for any $\epsilon > 0$ we can find a $\delta > 0$ such that if you move at most δ away from c to x in either direction, $|\varphi(x) - f(x)| < \epsilon|x - c|$.

Let $\epsilon = \frac{1}{2}$. No matter what line of positive slope we pick, the purple arrow representing the distance between $f(x)$ and $\varphi(x)$ is longer than the height of the region between the blue lines at x. A similar problem exists for lines of zero or negative slope, so we are able to say that the function is not differentiable at $x = 2$.

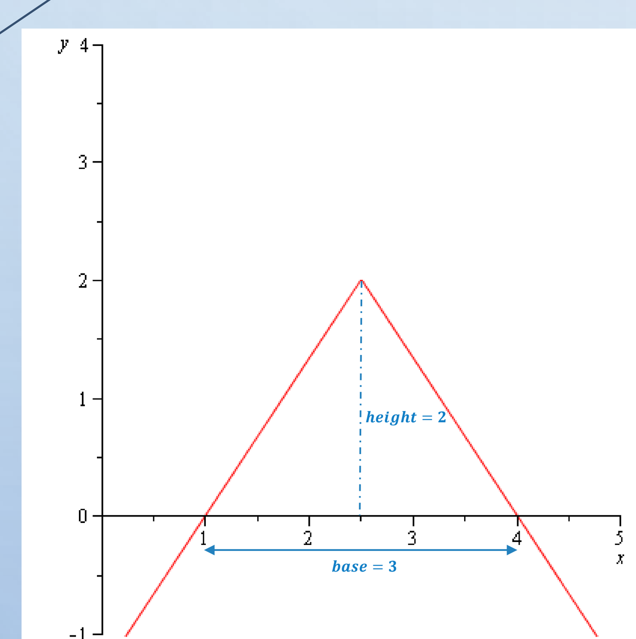


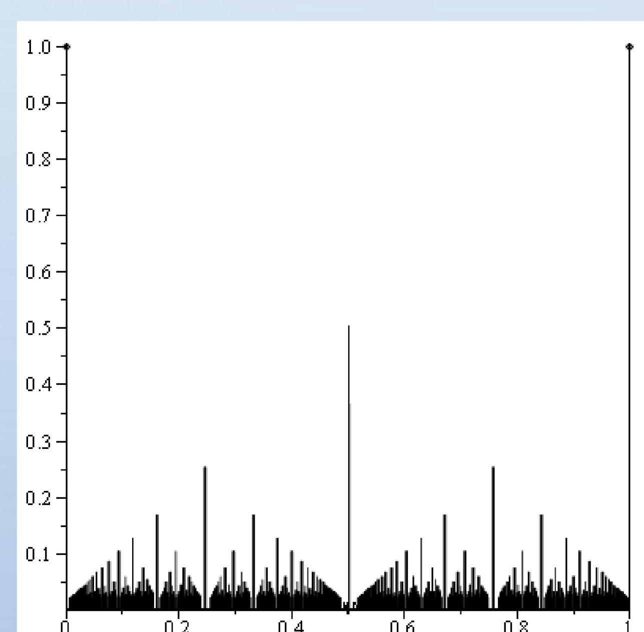
What About the Raindrop Function?

Since we can see that the raindrop function is not continuous at the rational numbers, we do not need to investigate its differentiability at these values because continuity is essential in order to be differentiable. At the irrational numbers we can use $\varphi(x) = 0$ to check for differentiability (because the values of the irrationals remain at zero). While the values of outputs of the raindrop function at the rational numbers come very close to zero as we saw with continuity, they do not come within a fraction of any line $\varphi(x)$.

Integrability

Can you find the area under the curve?





More Precisely:

We use what are called Riemann sums in order to determine integrability. We create a partition horizontally with varying widths and make rectangles by choosing the height of each rectangle equal to the value of the function at a point in the interval. If we can make an upper sum (the area of the tallest rectangles possible) and a lower sum (the area of the shortest rectangles possible) arbitrarily close, the function is called Riemann integrable.

With the raindrop function all lower sums are zero since there is an irrational number in any interval and the output of any irrational is zero. We can obtain an arbitrarily small upper sum by creating rectangles of extremely tiny widths around points where the function's value is large and larger widths around points where the value is close to zero so that the area of each of these rectangles is essentially zero. This means that the raindrop function is integrable and the integral is zero.

$$\textit{Area of a triangle} = \frac{1}{2}(\textit{base} * \textit{height}) \Rightarrow \int_1^4 f(x)dx = \frac{1}{2}(3 * 2) = 3$$