More on speedups of ergodic $\mathbb{Z}^d$-actions

David M. McClendon

Ferris State University
Big Rapids, MI, USA

joint with Aimee S.A. Johnson (Swarthmore)

Some new notation

Definition

Let $\mathbf{C} \subseteq \mathbb{Z}^d$ be any cone, and let $\mathbf{T}$ and $\mathbf{S}$ be m.p. $\mathbb{Z}^d$-actions. We say $\mathbf{T} \overset{\mathbf{C}}{\rightsquigarrow} \mathbf{S}$ if there is a $\mathbf{C}$-speedup of $\mathbf{T}$ which is isomorphic to $\mathbf{S}$.

Definition

Let $\mathbf{C} \subseteq \mathbb{Z}^d$ be any cone, and let $\mathbf{T}^\sigma$ and $\mathbf{S}^\sigma$ be G-extensions of m.p. $\mathbb{Z}^d$-actions. We say $\mathbf{T}^\sigma \underset{rel}{\overset{\mathbf{C}}{\rightsquigarrow}} \mathbf{S}^\sigma$ if there is a relative $\mathbf{C}$-speedup of $\mathbf{T}^\sigma$ which is relatively isomorphic to $\mathbf{S}^\sigma$.

Results about group extensions

in dimension 1:

Theorem (Arnoux, Ornstein & Weiss 1984)

If T is ergodic, and S is aperiodic, then $T \rightsquigarrow S$.

Theorem (Babichev, Burton & Fieldsteel 2013)

If T^σ (a G-extension) is ergodic and S (the base of some other G-extension) is aperiodic, then $T^\sigma \underset{rel}{\rightsquigarrow} S^\sigma$.

in dimension d:

Theorem 1 (Johnson-M)

If $\mathbf{T}^\sigma$ (a G-extension) is ergodic and $\mathbf{S}$ (the base of some other G-extension) is aperiodic, then for any cone $\mathbf{C}$, $\mathbf{T}^\sigma \overset{\mathbf{C}}{\underset{rel}{\rightsquigarrow}} \mathbf{S}^\sigma$.

Finite extensions

Notation: S_n is the symmetric group on n letters, which we will think of as acting on the finite set $[n] = \{1, 2, 3, ..., n\}$.
δ_n is uniform measure on the finite set $[n]$ (i.e. $\delta_n(x) = \frac{1}{n}$ for all x).

Definition

Let $(X, \mathcal{X}, \mu, \mathbf{T})$ be a m.p. system. A *n-point extension* of $\mathbf{T}$, a.k.a. *finite extension*, is a m.p. system $(X \times [n], \mathcal{X} \times 2^{[n]}, \mu \times \delta_n, \widetilde{\mathbf{T}}^\sigma)$ defined by

$$\widetilde{\mathbf{T}}^\sigma_{\mathbf{v}}(x, i) = (\mathbf{T}_{\mathbf{v}} x, \sigma(x, \mathbf{v}) i)$$

where σ is a cocycle taking values in S_n.
We call $\mathbf{T}$ the *base factor* of $\widetilde{\mathbf{T}}^\sigma$.

Finite extensions

Every finite extension $\widetilde{\mathbf{T}}^\sigma$ of $\mathbf{T}$ comes from a cocycle σ taking values in S_n.

$$\widetilde{\mathbf{T}}^\sigma_{\mathbf{v}}(x,i) = (\mathbf{T}_{\mathbf{v}}x, \sigma(x,\mathbf{v})i) \qquad (i \in [n])$$

Using σ to define an S_n-extension of $\mathbf{T}$, we obtain a group extension $\mathbf{T}^\sigma$ of $\mathbf{T}$ called the *full extension* of $\widetilde{\mathbf{T}}^\sigma$.

$$\mathbf{T}^\sigma_{\mathbf{v}}(x,g) = (\mathbf{T}_{\mathbf{v}}x, \sigma(x,\mathbf{v})g) \qquad (g \in S_n)$$

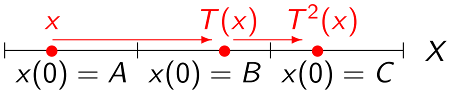
Finite extensions

Example

Let $(X, \mathcal{X}, \mu, T)$ be the full 3-shift (with alphabet A, B, C). Define $\sigma : X \to S_3$ by

$$\sigma(x) = \begin{cases} id & \text{if } x(0) = A \\ (123) & \text{if } x(0) = B \\ (132) & \text{if } x(0) = C \end{cases}$$

$x = \ldots ABC \ldots$



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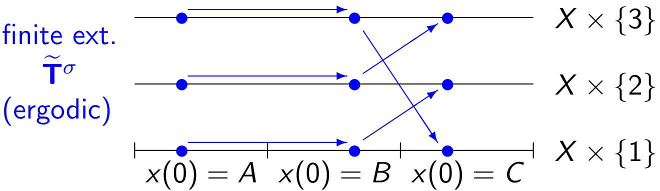

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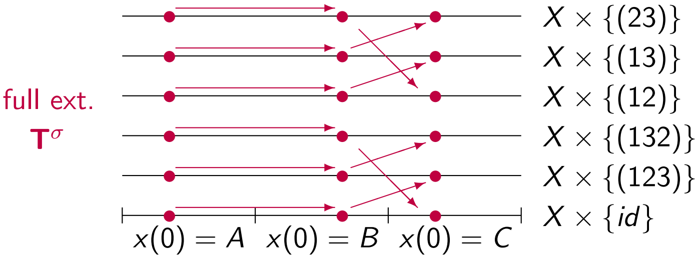

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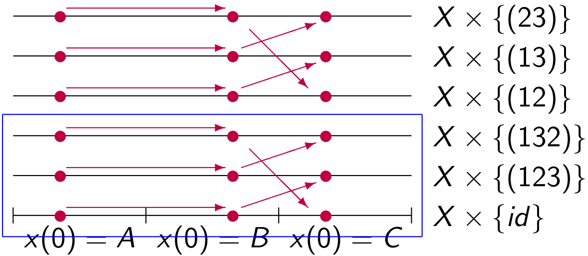
Note: $\mathbf{T}^\sigma$ is not ergodic



Speedups of finite extensions

Definition

Let $\mathbf{C} \subseteq \mathbb{Z}^d$ be any cone, and let $\widetilde{\mathbf{T}}^\sigma$ and $\widetilde{\mathbf{S}}^\sigma$ be *n*-point extensions. We say $\widetilde{\mathbf{T}}^\sigma \underset{rel}{\overset{\mathbf{C}}{\rightsquigarrow}} \widetilde{\mathbf{S}}^\sigma$ if there is a relative $\mathbf{C}$-speedup of $\widetilde{\mathbf{T}}^\sigma$ which is relatively isomorphic to $\widetilde{\mathbf{S}}^\sigma$.

Question

Under what circumstances does $\widetilde{\mathbf{T}}^\sigma \underset{rel}{\overset{\mathbf{C}}{\rightsquigarrow}} \widetilde{\mathbf{S}}^\sigma$?

Speedups of finite extensions

Idea: Given $\widetilde{\mathbf{T}}^\sigma$ and $\widetilde{\mathbf{S}}^\sigma$, let $\mathbf{T}^\sigma$ and $\mathbf{S}^\sigma$ be the respective full extensions.

Then

$$\widetilde{\mathbf{T}}^\sigma \underset{rel}{\overset{\mathbf{C}}{\rightsquigarrow}} \widetilde{\mathbf{S}}^\sigma \Leftrightarrow \mathbf{T}^\sigma \underset{rel}{\overset{\mathbf{C}}{\rightsquigarrow}} \mathbf{S}^\sigma$$

(by using the same speedup function $\mathbf{v}$).

So if $\mathbf{T}^\sigma$ is ergodic, this is always possible by Theorem 1.

What happens if $\mathbf{T}^\sigma$ is not ergodic?

It depends on the structure of the ergodic components of $\mathbf{T}^\sigma$ and $\mathbf{S}^\sigma$. The reason is that you can make a system "less ergodic" when you speed it up, but not "more ergodic".

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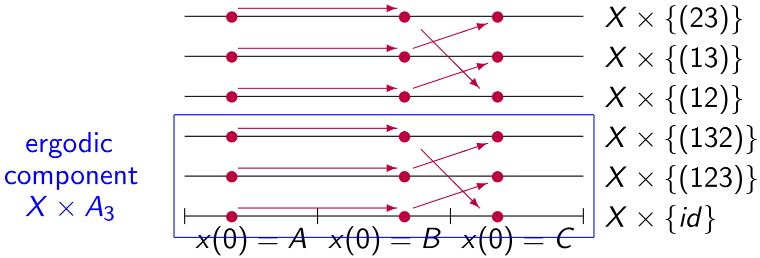
Speedups of finite extensions

Example (from before)

$$T \text{ is the full 3-shift; } \sigma(x) = \begin{cases} id & \text{if } x(0) = A \\ (123) & \text{if } x(0) = B \\ (132) & \text{if } x(0) = C \end{cases}$$

Recall that this 3-point extension was ergodic, but its full extension was not.



Bad news: In general, the full extension may not have such a simple ergodic decomposition.

Good news: Any full extension is relatively isomorphic to another S_n-extension which has $X \times G$ as one of its ergodic components, where G is some subgroup of S_n.

The set of possible Gs that can be obtained in this fashion form a conjugacy class of subgroups of S_n, and this class completely characterizes "speedupability".

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Speedups of finite extensions

Lemma

Let $\mathbf{T}$ be an ergodic $\mathbb{Z}^d$-action and let $\widetilde{\mathbf{T}}^\sigma$ be an n-point extension of $\mathbf{T}$. Then there is a conjugacy class $gp(\widetilde{\mathbf{T}}^\sigma) = gp(\mathbf{T}, \sigma)$ of subgroups of S_n such that TFAE:

1. $G \in gp(\widetilde{\mathbf{T}}^\sigma)$;
2. $\widetilde{\mathbf{T}}^\sigma$ is rel. isomorphic to some other n-point extension $\widetilde{\mathbf{T}}^{\sigma'}$ of $\mathbf{T}$ such that $X \times G$ is an ergodic component of the full extension of $\widetilde{\mathbf{T}}^{\sigma'}$.

$gp(\widetilde{\mathbf{T}}^\sigma)$ is called the *interchange class* of $\widetilde{\mathbf{T}}^\sigma$.

(Versions of this statement can be found in earlier work of Mackey, Zimmer, Rudolph, Gerber, perhaps others...)

Speedups of finite extensions

Theorem 2 ($d = 1$ Babichev, Burton & Fieldsteel 2013; $d > 1$ Johnson-M)

Let $\widetilde{\mathbf{T}}^\sigma$ and $\widetilde{\mathbf{S}}^\sigma$ be n-point extensions of ergodic $\mathbb{Z}^d$-actions $\mathbf{T}$ and $\mathbf{S}$, respectively. Then TFAE:

1. $\widetilde{\mathbf{T}}^\sigma \underset{rel}{\overset{\mathbf{C}}{\rightsquigarrow}} \widetilde{\mathbf{S}}^\sigma$;
2. For every $G_{\mathbf{T}} \in gp(\widetilde{\mathbf{T}}^\sigma)$, there is $G_{\mathbf{S}} \in gp(\widetilde{\mathbf{S}}^\sigma)$ such that $G_{\mathbf{S}} \subseteq G_{\mathbf{T}}$;
3. For some $G_{\mathbf{T}} \in gp(\widetilde{\mathbf{T}}^\sigma)$, there is $G_{\mathbf{S}} \in gp(\widetilde{\mathbf{S}}^\sigma)$ such that $G_{\mathbf{S}} \subseteq G_{\mathbf{T}}$.

Idea of proof (of $3 \Rightarrow 1$): Suppose $G_{\mathbf{T}} \in gp(\widetilde{\mathbf{T}}^\sigma)$; $G_{\mathbf{S}} \in gp(\widetilde{\mathbf{S}}^\sigma)$; $G_{\mathbf{S}} \subseteq G_{\mathbf{T}}$.

WLOG the full extension of $\widetilde{\mathbf{T}}^\sigma$ has ergodic component $X \times G_{\mathbf{T}}$.

Construct a relative speedup on this ergodic component so that $X \times G_{\mathbf{S}}$ is an ergodic component of the speedup (easy when $d = 1$: take first return map to $X \times G_{\mathbf{S}}$; not so easy when $d > 1$).

Use Theorem 1 to speed up this speedup (restricted to its ergodic component $X \times G_{\mathbf{S}}$) to obtain a isomorphic copy of the restriction of the full extension of $\widetilde{\mathbf{S}}^\sigma$ to $Y \times G_{\mathbf{S}}$. Mimic this construction (performed on the full extensions) on the finite extensions to prove the result.

Some examples

In the rest of this talk we will be considering two examples of two-point extensions.

Let τ denote the transposition $(1\,2)$, so that $S_2 = \{id, \tau\}$.

Notice that for any S_2-valued cocycle σ, since $gp(\mathbf{T}, \sigma)$ is a conjugacy class of subgroups of S_2, we have either that

$$gp(\mathbf{T}, \sigma) = \{id\} \quad \text{or} \quad gp(\mathbf{T}, \sigma) = \{S_2\}.$$

The first case corresponds exactly to when the full extension $\mathbf{T}^\sigma$ is not ergodic, and the second case corresponds to when the full extension $\mathbf{T}^\sigma$ is ergodic.

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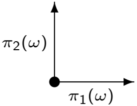
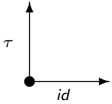
Example 1

Let $\Omega = S_2 \times S_2 = \{(id, id), (id, \tau), (\tau, id), (\tau, \tau)\}$. Let π_1 and π_2 be projections of the alphabet Ω onto the first and second coordinates, respectively.

Picture the elements of Ω this way:

$\omega \leftrightarrow$ 

e.g. $(id, \tau) \leftrightarrow$ 

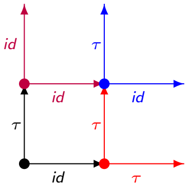
Example 1

Consider the $\mathbb{Z}^2$-SFT $\mathbf{S}$ with alphabet Ω where we only allow arrays $\{y_{\mathbf{v}} : \mathbf{v} \in \mathbb{Z}^2\}$ which satisfy, for every $\mathbf{v} = (v_1, v_2) \in \mathbb{Z}^2$,

$$\pi_2(y_{\mathbf{v}+(1,0)})\pi_1(y_{\mathbf{v}}) = \pi_1(y_{\mathbf{v}+(0,1)})\pi_2(y_{\mathbf{v}}).$$

This means we are only allowing arrays where the arrows form commutative diagrams.

legal:



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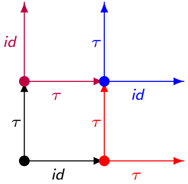


Example 1

Now define a two-point extension of $\mathbf{S}$ by the cocycle σ which is described by setting

$$\sigma_1(y) = \pi_1(y_{(0,0)}) \quad \sigma_2(y) = \pi_2(y_{(0,0)})$$

and extending in the natural way.

It's not too hard to check that the full extension $\mathbf{S}^\sigma$ is totally ergodic (each one-dimensional direction is isomorphic to the full shift on Ω).

That means

$$gp(\mathbf{S}, \sigma) = \{S_2\}$$

and for any $\mathbf{v} \neq (0,0)$ in $\mathbb{Z}^2$,

$$gp(\mathbf{S}_\mathbf{v}, \sigma) = \{S_2\}.$$

Example 2

Consider the $\mathbb{Z}^2$-SFT with alphabet $\{1, 2, 3, 4\}$ where we forbid any two symbols of the same parity to be adjacent ("diagonally adjacent" does not count as "adjacent").

For instance, if the symbol at position $(0, 0)$ is 3, then the symbols at positions $(1, 0)$, $(0, 1)$, $(-1, 0)$ and $(0, -1)$ must all be either 2 or 4.

Let E denote the set of "evens", i.e. the set of allowable arrays in this SFT which have symbol 2 or 4 at the origin.

Call this system $\mathbf{T}$.

$\mathbf{T}$ is ergodic but not totally ergodic (for example, E is invariant under $\mathbf{T}_{(2,0)}$).

Example 2

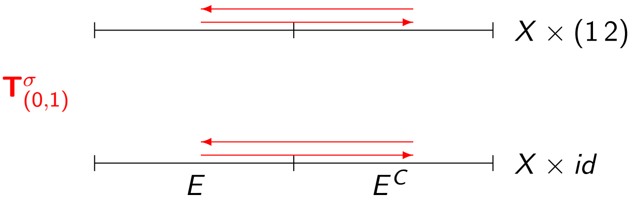
Now define a two-point extension of **T** via the cocycle

$$\sigma(x, (v_1, v_2)) = (1\,2)^{v_1}$$

(This extension "flips" points under the horizontal action, but not under the vertical action.)

Example 2

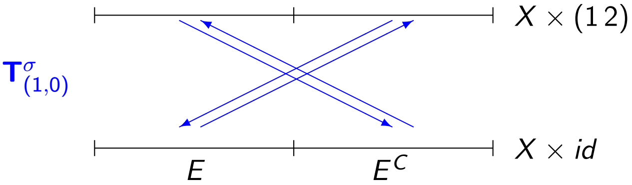
Recall $\sigma_{(v_1,v_2)}(x) = (1\,2)^{v_1}$.
Consider the action of $\mathbb{Z}$ generated by $\mathbf{T}^{\sigma}_{(0,1)}$:



Note: $X \times id$ is a nontrivial ergodic component of $\mathbf{T}^{\sigma}_{(0,1)}$. Therefore

$$gp(\mathbf{T}_{(0,1)}, \sigma) = \{id\}.$$

Recall $\sigma_{(v_1,v_2)}(x) = (1\,2)^{v_1}$.
Now, consider the action of $\mathbb{Z}$ generated by $\mathbf{T}^{\sigma}_{(1,0)}$:

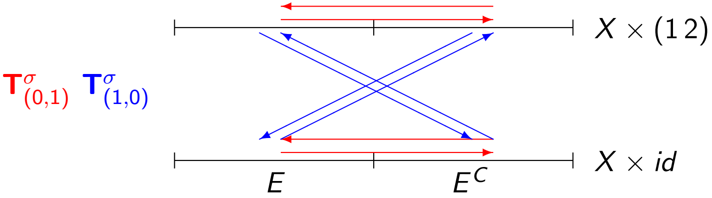


Note: $(E \times id) \cup (E^C \times (1\,2))$ is a nontrivial ergodic component of $\mathbf{T}^{\sigma}_{(1,0)}$. Therefore

$$gp(\mathbf{T}_{(1,0)}, \sigma) = \{id\}.$$

Example 2

Finally, consider the $\mathbb{Z}^2$-action $\mathbf{T}^\sigma$:



Note: $\mathbf{T}^\sigma$ is ergodic, so

$$gp(\mathbf{T}, \sigma) = \{S_2\}.$$

Comparing the two examples

We have a pair of two-point extensions, $\mathbf{S}^\sigma$ and $\mathbf{T}^\sigma$, with the following properties:

$$gp(\mathbf{S}, \sigma) = \{S_2\}$$

$$gp(\mathbf{S}_\mathbf{v}, \sigma) = \{S_2\} \quad \forall \mathbf{v} \neq (0,0)$$

$$gp(\mathbf{T}, \sigma) = \{S_2\}$$

$$gp(\mathbf{T}_{(0,1)}, \sigma) = \{id\}$$

$$gp(\mathbf{T}_{(1,0)}, \sigma) = \{id\}$$

Comparing the two examples

By Theorem 2, this means that for any cone $\mathbf{C}$,

$$\widetilde{\mathbf{T}}^{\sigma} \overset{\mathbf{C}}{\underset{rel}{\rightsquigarrow}} \widetilde{\mathbf{S}}^{\sigma}$$

but for any $\mathbf{v} \neq (0,0)$, it is **not** the case that

$$\widetilde{\mathbf{T}}^{\sigma}_{(0,1)} \underset{rel}{\rightsquigarrow} \widetilde{\mathbf{S}}^{\sigma}_{\mathbf{v}}$$

or

$$\widetilde{\mathbf{T}}^{\sigma}_{(1,0)} \underset{rel}{\rightsquigarrow} \widetilde{\mathbf{S}}^{\sigma}_{\mathbf{v}}.$$

In fact, one can show that no one-dimensional sub-action of this $\widetilde{\mathbf{T}}^{\sigma}$ can be relatively sped up to look like any one-dimensional sub-action of this $\widetilde{\mathbf{S}}^{\sigma}$.