Bounded speedups of Toeplitz systems

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Outline

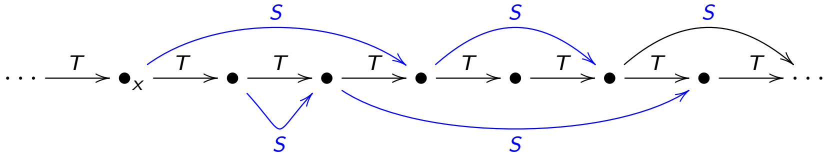
1. Definitions, history and context
2. Result advertised in the abstract: bounded speedups of Toeplitz flows need not be Toeplitz
3. Differences between our approach and that of Alvin-Radinger, who have independently proven the same result

Speedups

Definition

Given a dynamical system (X, T), a **speedup** of (X, T) is another system (X, S) where $S(x) = T^{p(x)}(x)$ for some function $p : X \to \mathbb{N}$.

p is called the **jump function** of the speedup.



Here, $p(x) = 3$, $p(T(x)) = 1$, etc.

What I mean by a “dynamical system”

In this talk, the (X, T) under consideration fall into two types:

Definition

A **measure-preserving system (m.p.s.)** is a triple (X, μ, T) where (X, μ) is a Lebesgue probability space and $T : X \to X$ is a measurable (a.s. $1-1$) transformation preserving μ.

Definition

A **Cantor minimal system (C.m.s.)** is a pair (X, T) where X is a Cantor space and $T : X \to X$ is a minimal homeomorphism.

The big picture

Question

Given two systems (X, T) and (Y, S), does there exist a speedup of T that is isomorphic to S?

Here, the word **isomorphic** means:

- measurably conjugate, if T and S are measure-preserving systems;
- topologically conjugate, if T and S are Cantor minimal systems.

Notation

Write

$$T \rightsquigarrow S$$

if there exists a speedup of T that is isomorphic to S.

The big picture

Question

Given two systems (X, T) and (Y, S), does there exist a speedup of T that is isomorphic to S?

Here, the word **isomorphic** means:

- measurably conjugate, if T and S are measure-preserving systems;
- topologically conjugate, if T and S are Cantor minimal systems.

Notation

For any adjective describing some class of functions on X, write

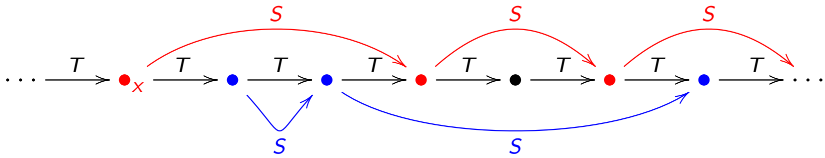
$$T \overset{\text{adjective}}{\rightsquigarrow} S$$

if $T \rightsquigarrow S$ via a jump function p which is that adjective.

Big picture idea

When T and S are orbit equivalent, we think of T and S as "having the same orbits".

When $T \rightsquigarrow S$, each T-orbit is partitioned into distinct S-orbits.



This suggests that the "speedup relation" $\rightsquigarrow$ has something to do with orbit equivalence, particularly if no further restrictions are placed on the jump function p.

History of speedups: ergodic theory

Theorem (Dye 1963)

Suppose T and S are ergodic m.p.t.s. Then T and S are (measurably) orbit equivalent.

Theorem (Arnoux-Ornstein-Weiss 1985)

Suppose T and S are m.p.t.s, where T is ergodic. Then $T \overset{\text{mble}}{\rightsquigarrow} S$.

Theorem (Connes-Feldman-Weiss 1981)

Suppose **T** and **S** are ergodic m.p. actions of $\mathbb{Z}^d$. Then **T** and **S** are measurably orbit equivalent.

Theorem (Johnson-M, 2014)

Suppose **T** and **S** are m.p. actions of $\mathbb{Z}^d$, where **T** is ergodic. Then $\mathbf{T} \overset{\text{mble}}{\rightsquigarrow} \mathbf{S}$.

History of speedups: ergodic theory

Theorem (Fieldsteel 1981)

Suppose $(X \times G, \widehat{T})$ and $(Y \times G, \widehat{S})$ are ergodic locally compact group extensions of m.p.t.s (X, T) and (Y, S). Then $(X \times G, \widehat{T})$ and $(Y \times G, \widehat{S})$ are relatively orbit equivalent (meaning orbit equivalent via a map measurable with respect to the factors (X, T) and (Y, S)).

Theorem (Burton-Babichev-Fieldsteel 2013)

Suppose $(X \times G, \widehat{T})$ and $(Y \times G, \widehat{S})$ are locally compact group extensions of m.p.t.s (X, T) and (Y, S), where $\widehat{T}$ is ergodic. Then $\widehat{T} \overset{\text{rel}}{\rightsquigarrow} \widehat{S}$.

Theorem (Giordano-Putnam-Skau 1995)

Let T and S be two C.m.s. TFAE:

1. T and S are topologically orbit equivalent.
2. T and S have isomorphic dimension groups.

Theorem (Ash-Ormes 2024)

Let T and S be two C.m.s. TFAE:

1. $T \overset{\text{lsc}}{\rightsquigarrow} S$.
   (“lsc” is “lower semicontinuous”)
2. There is a surjective group homomorphism from the dimension group of S to the dimension group of T, preserving the positive cones and distinguished order units of those groups.

History of speedups: topological dynamics

Theorem (Giordano-Matui-Putnam-Skau 2010)

Let $\mathbf{T}$ and $\mathbf{S}$ be two Cantor minimal $\mathbb{Z}^d$-actions. TFAE:

1. $\mathbf{T}$ and $\mathbf{S}$ are topologically orbit equivalent.
2. $\mathbf{T}$ and $\mathbf{S}$ have isomorphic dimension groups.

Theorem (Johnson-M 2022)

Let $\mathbf{T}$ and $\mathbf{S}$ be two Cantor minimal $\mathbb{Z}^d$-actions. TFAE:

1. $\mathbf{T} \overset{\text{Borel}}{\rightsquigarrow} \mathbf{S}$.
2. There is a surjective group homomorphism from the dimension group of $\mathbf{S}$ to the dimension group of $\mathbf{T}$, preserving the positive cones and distinguished order units of those groups.

History of speedups: topological dynamics

Theorem (Melleray 2019)

There exist two C.m.s. T and S such that

1. $T \rightsquigarrow S$ and
2. $S \rightsquigarrow T$, but
3. T and S are not topologically orbit equivalent.

Placing restrictions on the jump function

So far we have discussed results when the jump function p has minimal restrictions on it. These are generally related to orbit equivalence.

Placing restrictions on the jump function

When one starts to place requirements on p, however, the situation becomes more interesting.

In particular, we obtain classes of systems (in both the m.p. and C.m.s. categories) that are "closed under the taking of certain kinds of speedups".

Placing restrictions on the jump function

Natural requirements on p:

- We might require that p is bounded.
- We might require that p is integrable (in the m.p. category).
- We might require that p is measurable with respect to a particular factor (in the m.p. category).
- We might require that p is continuous (in the C.m.s. category).

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- We might require that p is measurable with respect to a particular factor (in the m.p. category).
- We might require that p is continuous (in the C.m.s. category).

For C.m.s., asking that p is continuous is equivalent to asking that p is bounded.

Placing restrictions on the jump function

Theorem (Neveu 1969)

Suppose (X, T) and (Y, S) are m.p.t.s. If $T \overset{\text{integrable}}{\rightsquigarrow} S$, then

$$\frac{h_\mu(S)}{h_\mu(T)}$$

is a positive whole number.

Two consequences:

1. There is no integrable speedup of the full 2-shift that is measurably conjugate to the full 3-shift.
2. Zero entropy m.p.t.s are "closed" under the taking of integrable speedups.

History of bounded speedups in topological dynamics

Alvin-Ash-Ormes (2018) studied what happens with speedups of C.m.s. when p is continuous (equivalently, bounded) and found several classes of systems closed under such speedups:

- Systems with zero topological entropy
- Odometers
- Expansive maps (a.k.a. subshifts)
- Systems of finite topological rank
- Substitutions

Odometers (more precisely, $\mathbb{Z}$-odometers)

Definition

Let $\mathfrak{G} = \{g_1, g_2, g_3, ...\}$ be an increasing sequence of whole numbers with $g_k \mid g_{k+1}$ for all k.

For each k, there is a quotient map

$$q_k : \mathbb{Z}/g_{k+1}\mathbb{Z} \to \mathbb{Z}/g_k\mathbb{Z}.$$

Let $X_{\mathfrak{G}} = \varprojlim (\mathbb{Z}/g_k\mathbb{Z})$, so a point $\mathbf{x} \in X_{\mathfrak{G}}$ formally looks like

$$\mathbf{x} = (x_1 + g_1\mathbb{Z}, x_2 + g_2\mathbb{Z}, x_3 + g_3\mathbb{Z}, ...)$$

where $q_k(x_{k+1} + g_{k+1}\mathbb{Z}) = x_k + g_k\mathbb{Z}$ for all k.

Definition (continued)

Define $\sigma_{\mathfrak{G}} : X_{\mathfrak{G}} \to X_{\mathfrak{G}}$ by

$$\sigma_{\mathfrak{G}}(\mathbf{x}) = (x_1 + 1 + g_1\mathbb{Z}, x_2 + 1 + g_2\mathbb{Z}, ...).$$

Any action conjugate to such a $(X_{\mathfrak{G}}, \sigma_{\mathfrak{G}})$ is called an **($\mathbb{Z}$-)odometer**.

The sequence $\mathfrak{G} = \{g_1, g_2, g_3, ...\}$ is called a **scale** for $(X_{\mathfrak{G}}, \sigma_{\mathfrak{G}})$.

Odometers (more precisely, $\mathbb{Z}$-odometers)

Remarks

If the scale $\mathfrak{G} = \{g_k\}$ is eventually constant, then $\#(X_\mathfrak{G})$ is finite, and $\sigma_\mathfrak{G}$ is the action of $\mathbb{Z}$ by translations on some finite cyclic group $\mathbb{Z}/g_K\mathbb{Z}$. We call such an action a **finite odometer** and denote it by $\tau_{\mathbb{Z}/g_K\mathbb{Z}}$.

Otherwise, we say $\sigma_\mathfrak{G}$ is an **infinite odometer**; in this case $(X_\mathfrak{G}, \sigma_\mathfrak{G})$ is a C.m.s.

Every odometer is uniquely ergodic with respect to the Haar measure on $X_\mathfrak{G}$.

Odometers (more precisely, $\mathbb{Z}$-odometers)

Another remark

The odometer $(X_{\mathfrak{G}}, \sigma_{\mathfrak{G}})$ given by scale $\mathfrak{G} = \{g_1, g_2, g_3, ...\}$ factors onto each of the finite odometers $(\mathbb{Z}/g_k\mathbb{Z}, \tau_{\mathbb{Z}/g_k\mathbb{Z}})$ by projecting each $\mathbf{x} \in X_{\mathfrak{G}}$ onto its k^{th} coordinate.

So infinite odometers have *lots of finite factors*, and therefore *lots of rational eigenvalues*.

As an example, take the dyadic odometer given by $g_k = 2^k$. For every $\lambda \in \mathbb{Z}\left[\frac{1}{2}\right]$, $e^{2\pi i \lambda}$ is an eigenvalue of $\sigma_{\mathfrak{G}}$, i.e. there is a continuous function $f : X_{\mathfrak{G}} \to \mathbb{C}$ with

$$f \circ \sigma_{\mathfrak{G}} = e^{2\pi i \lambda} f.$$

Odometers (more precisely, $\mathbb{Z}$-odometers)

Theorem (Alvin-Ash-Ormes 2018)

Let T be a $\mathbb{Z}$-odometer, and suppose $T \overset{\text{cts}}{\rightsquigarrow} S$. If S is minimal, then S is a $\mathbb{Z}$-odometer which is topologically conjugate to T.

Note: For actions of groups other than $\mathbb{Z}$, part of this theorem doesn't hold: a continuous speedup $\mathbf{S}$ of a $\mathbb{Z}^d$-odometer $\mathbf{T}$ must be a $\mathbb{Z}^d$-odometer, but $\mathbf{S}$ and $\mathbf{T}$ need only be orbit equivalent (and not necessarily conjugate).

Substitution systems

Definition

Let $\mathcal{A}$ be a finite alphabet and let $\mathcal{A}^+$ be the set of (finite-length) words with letters taken from $\mathcal{A}$. A **substitution (map)** is a function $\theta : \mathcal{A} \to \mathcal{A}^+$.

Ex: $\mathcal{A} = \{a, b\}$, $\theta(a) = aab$, $\theta(b) = abab$

Extend θ by concatenation to obtain $\theta : \mathcal{A}^+ \to \mathcal{A}^+$ (also $\theta : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$).

Ex: θ as above $\Rightarrow \theta(abba) = aababababaab$

Iterate θ to obtain $\theta^n : \mathcal{A} \to \mathcal{A}^+$ for each n (and extend by concatenation to define θ^n on $\mathcal{A}^+$ and $\mathcal{A}^{\mathbb{Z}}$).

Ex: $\theta^2(a) = \theta(aab) = aabaababab$

Definition

A substitution θ is called ...

- ... **primitive** if for every $a, b \in \mathcal{A}$, $\exists\, n \geq 1$ so that $\theta^n(a)$ contains the symbol b, and if for every $a \in \mathcal{A}$,

$$\text{length of } \theta^n(a) \stackrel{n\to\infty}{\longrightarrow} \infty.$$

- ... **proper** if $\exists\, l, r \in \mathcal{A}$ and $n \geq 1$ so that for every $a \in \mathcal{A}$, $\theta^n(a) = l......r$.
- ... **constant length** if the length of $\theta(a)$ is the same for every $a \in \mathcal{A}$.

Definition

Given a substitution map θ, let $X_\theta \subseteq \mathcal{A}^{\mathbb{Z}}$ be the set of sequences $\mathbf{x} = ...x_0x_1x_2....$ so that for every $i < j$, the word $x_ix_{i+1}...x_j$ is a subword of $\theta^n(a)$ for some $a \in \mathcal{A}$.

X_θ is invariant under the shift σ, so this yields a dynamical system (X_θ, σ) called the **substitution system (generated by θ)**.

θ is called **aperiodic** if X_θ contains no periodic points.

Substitution systems

Theorem (Queffelec 1987)

If θ is an aperiodic, primitive substitution, then (X_θ, σ) is a C.m.s.

Theorem (Durand-Host-Skau 1999)

Every substitution system is conjugate to one generated by a proper substitution.

Furthermore, in this case there is a unique element of $\mathcal{A}^{\mathbb{Z}}$ fixed under θ, and X_θ is the orbit closure of this element under the shift.

Substitution systems

Theorem (Alvin-Ash-Ormes 2018)

Let (X_θ, σ) be a minimal substitution system coming from a proper, primitive substitution θ and suppose $T \overset{\text{cts}}{\rightsquigarrow} S$. If S is minimal, then (X_θ, S) is a substitution system.

In the same paper, Alvin-Ash-Ormes gave necessary and sufficient conditions for when a bounded speedup of a substitution system is minimal.

They use what they call "orbit block labeling permutations" to define a different substitution on a larger alphabet; this new substitution is primitive if and only if the speedup is minimal.

Substitution systems

Ash-Dykstra-LeMasurier (2023) took the Alvin-Ash-Ormes machinery and used it to describe an explicit substitution rule for any minimal, continuous speedup of a minimal substitution system, in terms of the original substitution and the jump function.

They did this by carefully analyzing how Kakutani-Rohklin partitions associated to the original substitution are "split" into Kakutani-Rohklin partitions for the speedup, and translated this splitting into the language of Bratteli-Vershik diagrams.

Kakutani-Rohklin partitions

Definition

A **Kakutani-Rohklin (K-R) partition** for C.m.s. (X, T) is a partition of X of the form

$$\mathcal{P} = \{T^j(B^i) : 1 \le i \le t, 0 \le j < h^i\}.$$

where each B^i is clopen.
This partition divides X into t **towers**, where the i^{th} tower of $\mathcal{P}$ is

$$\{T^j(B^i) : 0 \le j < h^i\}.$$

B^i is called the **base** of this tower; the **height** of this tower is h^i.

Kakutani-Rohklin partitions

Theorem (Herman-Putnam-Skau 1992)

Every C.m.s. (X, T) admits a sequence $\{P(n)\}$ of K-R partitions satisfying these four conditions:

(KR1) The bases of $\{P(n)\}$ form a nested decreasing sequence of sets;

(KR2) Each level of $\mathcal{P}(n+1)$ is contained in a single level of $\mathcal{P}(n)$;

(KR3) The intersection of the bases of all the towers in all the $\mathcal{P}(n)$ is a single point in X;

(KR4) The partitions $\mathcal{P}(n)$ generate the topology of X.

Kakutani-Rohklin partitions

Any sequence $\{\mathcal{P}(n)\}$ of K-R partitions satisfying (KR1)-(KR4) can be translated into a properly ordered Bratteli diagram.

(KR1) and (KR2) ensure that the towers in each partition is obtained by "cutting and stacking" the towers of the previous partition.

(KR3) ensures that the associated Bratteli diagram is properly ordered (has unique maximal and minimal paths).

(KR4) guarantees a conjugacy between (X, T) and its Bratteli-Vershik model.

K-R partitions for substitution systems

Suppose (X_θ, σ) is a minimal substitution system over a finite alphabet $\mathcal{A}$. For each $n \in \mathbb{N}$ and each $i \in \mathcal{A}$, let

$$h^i = \text{length}(\theta^n(i))$$

and set

$$\begin{aligned} B^i(n) &= [\theta^n(i)] \\ &= \{\mathbf{x} \in X_\theta : x_j = (\theta^n(i))_j \text{ for all } j \in \{0, 1, ..., h^i - 1\}. \end{aligned}$$

Finally,

$$\mathcal{P}(n) = \bigsqcup_{i=1}^{\#(\mathcal{A})} \bigsqcup_{j=1}^{h^i} \sigma^j(B^i(n)).$$

K-R partitions for substitution systems

Theorem

Let (X_θ, σ) be a minimal substitution system.

The $\mathcal{P}(n)$ described on the previous slide form a sequence of K-R partitions for (X_θ, σ) satisfying (KR1)-(KR4).

Tower splitting: K-R partitions for speedups

Ash-Dykstra-LeMasurier (2023) start with a minimal substitution system (X_θ, T) which has a minimal bounded speedup (X_θ, S).

They construct the sequence $\mathcal{P}(n)$ of K-R partitions described on the previous slides.

Since the jump function p is continuous, there is some N so that p is constant on each level of $\mathcal{P}(N)$.

For each $n \geq N$ and each tower in $\mathcal{P}(n)$, they split each tower in $\mathcal{P}(n)$ into separate S-towers, yielding a K-R partition $\widetilde{\mathcal{P}}(n)$ for S.

Tower splitting: K-R partitions for speedups

Let $B = B^i(n)$ denote the base of the i^{th} tower $\mathcal{P}^i(n)$ which is part of the K-R partition $\mathcal{P}(n)$. If that tower has height $h = h^i(n)$, it looks like this:

$$\begin{array}{c} T^{h-1}(B) \\ \vdots \\ T^4(B) \\ T^3(B) \\ T^2(B) \\ T(B) \\ B \end{array}$$

$$\boxed{\mathcal{P}^i(n)}$$

Tower splitting: K-R partitions for speedups

$$T^{h-1}(B)$$

$$\vdots$$

$$T^8(B)$$

$$T^7(B)$$

$$T^6(B)$$

$$T^5(B)$$

$$T^4(B)$$

$$T^3(B)$$

$$T^2(B)$$

s $\quad T(B)$

$$B$$

$$\boxed{\mathcal{P}^i(n)}$$

Tower splitting: K-R partitions for speedups

$T^{h-1}(B)$

$\vdots$

$T^8(B)$

$T^7(B)$

$T^6(B)$

$T^5(B)$

$T^4(B)$

$T^3(B)$

s

$T^2(B)$

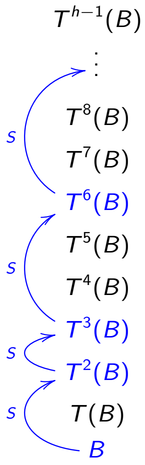
s

$T(B)$

B

$\boxed{\mathcal{P}^i(n)}$

Tower splitting: K-R partitions for speedups



$\boxed{\mathcal{P}^i(n)}$

$T^{h-1}(B)$

$\vdots$

$T^8(B)$

$T^7(B)$

S

$T^6(B) = S^3(\widetilde{B}_1)$

$T^5(B)$

S

$T^4(B)$

$T^3(B) = S^2(\widetilde{B}_1)$

S

$T^2(B) = S(\widetilde{B}_1)$

$T(B)$

S

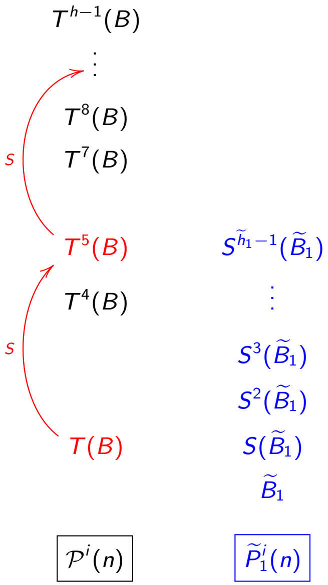
$B = \widetilde{B}_1$

$\boxed{\mathcal{P}^i(n)}$

Tower splitting: K-R partitions for speedups

$T^{h-1}(B)$	
$\vdots$	
$T^8(B)$	
$T^7(B)$	
$T^5(B)$	$S^{\widetilde{h}_1-1}(\widetilde{B}_1)$
$T^4(B)$	$\vdots$
	$S^3(\widetilde{B}_1)$
	$S^2(\widetilde{B}_1)$
$T(B)$	$S(\widetilde{B}_1)$
	$\widetilde{B}_1$
$\boxed{\mathcal{P}^i(n)}$	$\boxed{\widetilde{P}_1^i(n)}$

Tower splitting: K-R partitions for speedups



Tower splitting: K-R partitions for speedups

$$T^{h-1}(B)$$

$$\vdots$$

$$T^8(B)$$

$$T^7(B)$$

$$S^{\widetilde{h}_1-1}(\widetilde{B}_1)$$

$$T^4(B) \qquad \vdots \qquad S^{\widetilde{h}_2-1}(\widetilde{B}_2)$$

$$S^3(\widetilde{B}_1) \qquad \vdots$$

$$S^2(\widetilde{B}_1) \qquad S^2(\widetilde{B}_2)$$

$$S(\widetilde{B}_1) \qquad S(\widetilde{B}_2)$$

$$\widetilde{B}_1 \qquad \widetilde{B}_2$$

$$\boxed{\mathcal{P}^i(n)} \qquad \boxed{\widetilde{P}^i_1(n)} \qquad \boxed{\widetilde{P}^i_2(n)}$$

Continue in this fashion to "split" the T-tower $\mathcal{P}^i(n)$ into S-towers

$$\widetilde{\mathcal{P}}_1^i(n), \widetilde{\mathcal{P}}_2^i(n), ..., \widetilde{\mathcal{P}}_{\overline{p}}^i(n),$$

where each level of the S-towers is a level of $\mathcal{P}^i(n)$.

Tower splitting: K-R partitions for speedups

Repeating this procedure on every tower of $\mathcal{P}(n)$, you get a K-R partition for S:

$$\widetilde{\mathcal{P}}(n) = \bigsqcup_i \bigsqcup_j \widetilde{\mathcal{P}}^i_j(n) = \bigsqcup_i \bigsqcup_j \bigsqcup_{v=0}^{\widetilde{h}^i_j(n)-1} S^v(\widetilde{B}^i_n).$$

And the sequence $\{\widetilde{\mathcal{P}}(n)\}$ of K-R partitions for S satisfy (KR1), (KR2) and (KR4).

But: this sequence $\{\widetilde{P}(n)\}$ doesn't satisfy (KR3) (no unique maximal/minimal points).

Because of this, the Bratteli-Vershik representations they correspond to for the sped up substitutions are "abnormal".

Definition

Let (X, T) be a topological dynamical system. $x \in X$ is **regularly recurrent** if for every open $U \ni x$, the set $\{n : T^n(x) \in U\}$ contains an infinite arithmetic progression.

A **Toeplitz sequence** is a $\mathbf{w} \in \mathcal{A}^{\mathbb{Z}}$ which is regularly recurrent under the shift σ.

A **Toeplitz flow** is any C.m.s. topologically conjugate to $(X_{\mathbf{w}}, \sigma)$, where $X_{\mathbf{w}}$ is the orbit closure of a Toeplitz sequence $\mathbf{w}$.

Toeplitz flows are related to substitutions:

Theorem (Durand-Host-Skau 1999)

Every minimal substitution system generated by an aperiodic, proper, primitive substitution of constant length is a Toeplitz flow.

Toeplitz systems

Toeplitz flows can be characterized in many different ways:

Theorem (Gjerde-Johansen 2000)

A topological dynamical system (X, T) is a Toeplitz flow if and only if it is conjugate to the Vershik map on a properly ordered Bratteli diagram with the **equal path number property**, meaning that the number of paths between two vertices at different levels of the Bratteli diagram depends only on the levels those vertices are in.

This does not mean that *every* presentation of (X, T) via a Bratteli diagram must be one with the equal path number property, however.

Toeplitz systems

Theorem (stated in Downarowicz 2005)

A topological dynamical system (X, T) is conjugate to a Toeplitz flow if and only if it is a minimal symbolic dynamical system that is an almost $1-1$ extension of an odometer.

"Almost $1-1$" means that there exists a factor map $\pi : (X, T) \to (X_{\mathfrak{G}}, \sigma_{\mathfrak{G}})$ where, for a dense set of $\mathbf{x}$, $\pi^{-1}(\mathbf{x})$ is a singleton.

So since odometers have lots of finite factors and lots of rational eigenvalues, Toeplitz flows also have *lots of finite factors*, and therefore *lots of rational eigenvalues*.

Ash, Dykstra and LeMasurier knew that continuous speedups of odometers are odometers, and knew that Toeplitz flows are almost $1-1$ extensions of odometers. Together with data coming from an example they worked out in their paper, they conjectured:

Conjecture (Ash-Dykstra-LeMasurier 2023)

Let (X,T) be a Toeplitz flow and suppose $T \overset{\text{cts}}{\rightsquigarrow} S$. If S is minimal, then (X,S) is a Toeplitz flow.

However, this conjecture is rather strongly false.

Bounded speedups of Toeplitz systems need not be Toeplitz

Theorem (Johnson-M)

Let (X_θ, σ) be any system generated by a primitive, proper, aperiodic substitution of constant length (so (X_θ, T) is Toeplitz). There exists a minimal bounded speedup (X_θ, S) of (X_θ, σ) which is not Toeplitz.

Remark: As we may see later, our speedup S is more than just "not Toeplitz".

Remark: Independently, Alvin-Radinger have a different example of a bounded speedup of a Toeplitz flow which is not Toeplitz.

Bounded speedups of Toeplitz systems need not be Toeplitz

Theorem (Johnson-M)

Let (X_θ, σ) be any system generated by a primitive, proper, aperiodic substitution of constant length (so (X_θ, T) is Toeplitz). There exists a minimal bounded speedup (X_θ, S) of (X_θ, σ) which is not Toeplitz.

Remark: As we may see later, our speedup S is more than just "not Toeplitz".

Our example, and our argument as to why our speedup isn't Toeplitz, is different than that of Alvin-Radinger, and leads us to consider a different collection of associated problems.

Outline of proof

1. Choose a "special" letter of the alphabet $\mathcal{A}$ that has certain technical properties.
2. Take the sequence $\{\mathcal{P}(n)\}$ of K-R partitions for (X_θ, σ), where the base of each tower in $\mathcal{P}(n)$ is $[\theta^n(a)]$ for $a \in \mathcal{A}$.
3. Choose n large enough.
4. Define the jump function p level-by-level on each level of $\mathcal{P}(n)$; p is defined the same way on all the towers except the one associated to the special symbol chosen in Step 1.
5. Show that the definition of p effectively rigs the heights of the "split towers" and the "orbit labelling permutations" in a certain way.
6. Construct an associated speedup on a larger alphabet as in Alvin-Ash-Ormes; show this substitution is primitive, so our speedup is minimal (same argument as Alvin-Radinger example).
7. Prove the speedup is not Toeplitz (different argument from Alvin-Radinger).

Some motivation from ergodic theory

The inspiration for our family of examples and our method of proof is derived from prior work related to the classification of classifying measure-preserving transformations up to conjugacy.

Foreman, Rudolph and Weiss (2011) showed that the relation of conjugacy on the space of all such transformations is not Borel, but the relation of conjugacy on the set of all *(measurably) rank one* transformations is Borel.

Some motivation from ergodic theory

Definition

(X, μ, T) is **(measurably) rank one** if there is a sequence $\{\mathcal{T}_n\}$ of towers, so that for any $A \in \mathcal{X}$ and every n, there is a union L_n of levels of $\mathcal{T}_n$ so that

$$\lim_{n\to\infty} \mu(L_n \triangle A) = 0.$$

In other words, (X, T) can be (a.s.) constructed via cutting and stacking, where at each stage there is a single tower and an "error set", part of which gets inserted into the tower at the next stage as "spacers". The measure of the error set at stage n decays to 0 as $n \to \infty$.

Definition

Suppose (X, μ, T) is measurably rank one with associated sequence $\{\mathcal{T}_n\}$, and denote the base of $\mathcal{T}_n$ by B_n. For any $m \leq n$, set

$$I_{m,n} = \{v \in F_n : T^v(B_n) \subseteq B_m\}.$$

$I_{m,n}$ is called the **(set of) descendants** of (the base of) $\mathcal{T}_m$ in the tower $\mathcal{T}_n$.

In other words, the descendants are the heights of the levels in tower $\mathcal{T}_n$ which came from the base of tower $\mathcal{T}_m$.

Some motivation from ergodic theory

Theorem (Foreman-Gao-Hill-Silva-Weiss 2023)

Let (X, μ, T) be measurably rank one and let $\sigma_{\mathfrak{G}}$ be the $\mathbb{Z}$-odometer with scale $\mathfrak{G}$. TFAE:

1. (X, T) factors onto $(X_{\mathfrak{G}}, \sigma_{\mathfrak{G}})$.
2. "For every k in the scale, eventually most descendants are congruent mod k." This means: $\forall\, k \in \mathfrak{G}, \forall\, \epsilon > 0$, $\exists N \in \mathbb{N}$ so that $\forall n \geq m \geq N$, $\exists\, g \in \mathbb{Z}$ so that
$$\frac{\#(v \in I_{m,n} : v \not\equiv g \bmod k\})}{\#(I_{m,n})} < \epsilon.$$

Johnson-M (2024) extended this result to funny rank one $\mathbb{Z}$-actions and actions of more general groups.

A topological version of FGHSW

FGHSW give a criterion for when a measurably rank one system factors (measurably) onto an odometer.

Question 1 (dumb)

Can we characterize the topologically rank one systems that factor onto an odometer?

Preliminary question

What does it mean for a Cantor minimal system to be *topologically rank one*?

A topological version of FGHSW

Definition

A C.m.s. (X, T) has **finite topological rank** if it is conjugate to a Vershik map on a Bratelli diagram, where the number of vertices at each level of the diagram is uniformly bounded by some natural number s.

Equivalently, (X, T) has rank $\leq s$ if it has a sequence of K-R partitions satisfying (KR1)-(KR4) where each partition consists at most s towers.

Definition

(X, T) has **topological rank** r if r is the smallest value of s for which the previous sentence holds.

A topological version of FGHSW

With this definition in hand, it is easy to see this:

Theorem

The C.m.s. (X, T) has topological rank one if and only if (X, T) is an odometer.

So the answer to Question 1 is "all of them".

A topological version of FGHSW

Question 2

Can we characterize the Cantor minimal systems of finite rank that factor onto an odometer (based on descendants or similar information coming from K-R partitions)?

Why Cantor minimal systems <u>of finite rank</u>?

Theorem (Deka-(Garcia-Ramos)-Kasprzak-Kunde-Kwietniak 2025)

The conjugacy relation on Cantor minimal systems is not Borel.

A potential difficulty

As mentioned earlier, the sequence of K-R partitions for a speedup (X, S) described by ADL do not, in general, satisfy (KR3) (no unique maximal and/or minimal points).

This presents a problem when trying to apply previous results regarding Question 2 that use spectral arguments or just examine the heights of the towers at each level.

However, we are able to follow the ideas of FGHSW to obtain a topological version of their theorem that does not require that the K-R partitions in question satisfy (KR3).

Existence of an odometer factor

Definition

Let (X, T) be a finite rank C.m.s. with $\{\mathcal{P}(n)\}$ an associated sequence of K-R partitions satisfying (KR2) and (KR4). Let $m \leq n$.

For each $i_m \in \{1, ..., t(m)\}$ and each $i_n \in \{1, ..., t(n)\}$, let

$$I_{(m,i_m),(n,i_n)} = \{v \in F_n : T^v(B_{i_m}(n)) \subseteq B_{i_m}(m)\}.$$

$I_{(m,i_m),(n,i_n)}$ is called the set of **descendants** of the i_m^{th} tower of $\mathcal{P}(m)$ in the i_n^{th} tower of $\mathcal{P}(n)$.

These are the levels in the i_n^{th} stage n tower that come from the base of the i_m^{th} stage m tower.

Existence of an odometer factor

Theorem (Johnson-M)

Let (X, T) be a finite rank C.m.s. with $\{\mathcal{P}(n)\}$ an associated sequence of K-R partitions satisfying (KR2) and (KR4). Let $(X_{\mathfrak{G}}, \sigma_{\mathfrak{G}})$ be an odometer. TFAE:

1. $\exists$ topological factor map $\psi : (X, T) \to (X_{\mathfrak{G}}, \sigma_{\mathfrak{G}})$.
2. "For every k in the scale of the odometer, the descendants are eventually congruent mod k."
   This means that for all $k \in \mathfrak{G}$, there exists $N \in \mathbb{N}$ so that for all $n > N$ and all i_N, i_n, there is $g \in \mathbb{Z}$ so that $I_{(N,i_N),(n,i_n)} \subseteq g + k\mathbb{Z}$.

Existence of an odometer factor

Returning to the proof that our speedup of a Toeplitz flow need not be Toeplitz:

Using the "tower splitting" of Ash-Dykstra-LeMasurier, it is straightforward to compute (some of) the descendants for a set of K-R partitions (that don't satisfy (KR3)) associated to the speedup we construct. These descendants fail the condition of the previous theorem.

Thus our speedups have no odometer factor (*no finite factors* and *no continuous rational eigenvalues*), and therefore cannot be Toeplitz.

Conjugacy with an odometer

Foreman et al. also identified conditions under which a measurably rank one system is measurably *conjugate* to an odometer:

Conjugacy with an odometer

Theorem (Foreman-Gao-Hill-Silva-Weiss 2023)

Let (X, μ, T) be measurably rank one and let $\sigma_{\mathfrak{G}}$ be the $\mathbb{Z}$-odometer with scale $\mathfrak{G}$. TFAE:

1. (X, T) is measurably conjugate to $(X_{\mathfrak{G}}, \sigma_{\mathfrak{G}})$.
2. Both of the following statements hold:
    1. "For every k in the scale, eventually most descendants are congruent mod k."
    (same condition as their earlier theorem)
    2. "Eventually, the descendants of every partition mostly coincide with a subset of some $\mathbb{Z}/k\mathbb{Z}$, where k is from the scale."
    This means that for all $l \in \mathbb{N}$, all $\epsilon > 0$, there exists $k \in \mathfrak{G}$ and $N \in \mathbb{N}$ so that for all $m \geq N$, there is a set $D \subseteq \mathbb{Z}/k\mathbb{Z}$ so that

$$\frac{\#[(D \cap \{0, ..., h_m - 1\}) \,\Delta\, I_{l,m})]}{\#(I_{l,m})} < \epsilon.$$

Theorem (Johnson-M)

Let (X, T) be a finite rank C.m.s. and $\{\mathcal{P}(n)\}$ a sequence of K-R partitions satisfying (KR2) and (KR4). Let $(X_\mathfrak{G}, \sigma_\mathfrak{G})$ be an odometer. TFAE:

1. (X, T) is topologically conjugate to $(X_\mathfrak{G}, \sigma_\mathfrak{G})$.
2. Both of the following statements hold:
   1. "For every k in the scale, eventually all the descendants are congruent mod k."
      (same condition as our earlier theorem)
   2. "Eventually, the descendants of every partition exactly coincide with a subset of some $\mathbb{Z}/k\mathbb{Z}$, where k is from the scale."
      This means that for all $l \in \mathbb{N}$, there exists $k \in \mathfrak{G}$ and $N_a \in \mathbb{N}$ so that for all $m \geq N_a$ and all i, i', there is a set $D \subseteq \mathbb{Z}/k\mathbb{Z}$ so that

$$I_{(l,i),(m,i')} = D \cap \{0, ..., h^{i'}(m) - 1\}.$$