Rank one $\mathbb{Z}^d$-actions conjugate to odometers

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joint with Aimee S.A. Johnson (Swarthmore)

Big picture

Classification problem (proposed by von Neumann)

Classify (ergodic) m.p. actions up to conjugacy.

Classifying actions with discrete spectrum: solved (von Neumann proved eigenvalues are a complete invariant)

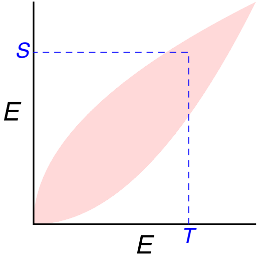
Classifying Bernoulli actions: solved (Ornstein proved entropy is a complete invariant)

Big picture

Classification problem (proposed by von Neumann)

Classify (ergodic) m.p. actions up to conjugacy.

Classifying all actions: intractable (Foreman-Rudolph-Weiss proved that conjugacy of ergodic m.p. systems is complete analytic)



(color (T, S) pink if T is conjugate to S; the pink region isn't Borel)

Big picture

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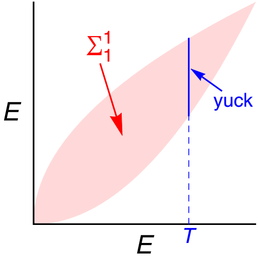
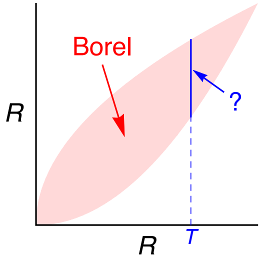


Classification problem (proposed by von Neumann)

Classify (ergodic) m.p. actions up to conjugacy.

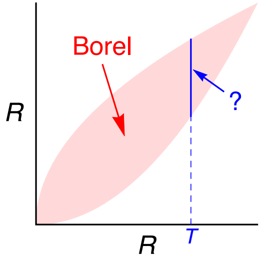
Classifying rank one actions: ? (Foreman-Rudolph-Weiss proved that conjugacy of rank one systems is Borel)



Big picture

Motivation

The work I am going to discuss deals with characterizing the set labelled with the "?" when T is an odometer.



Outline of the talk

1. Review *rank one* systems, setting up language for later
2. Review *odometers*, setting up language for later
3. State results describing the sets labelled earlier with "?"
4. Brief comments on some ideas in proofs (time permitting)

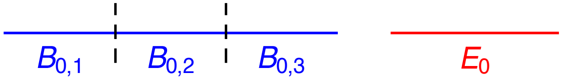
What is “rank one”?

From the table of contents of (Ferenczi 1997):

An example: Chacon map

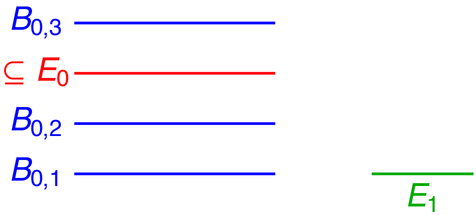
Stage 0: Start by partitioning X into B_0 and E_0:

Stage 0 $\rightarrow$ **Stage 1:** Cut and stack B_0, inserting spacer taken from E_0:



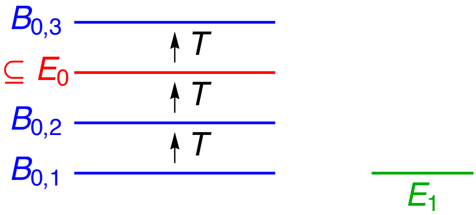
An example: Chacon map

Stage 0 $\rightarrow$ **Stage 1:** Cut and stack B_0, inserting spacer taken from E_0:



An example: Chacon map

Stage 0 $\to$ **Stage 1:** Cut and stack B_0, inserting spacer taken from E_0:



An example: Chacon map

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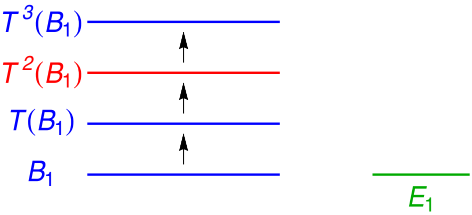
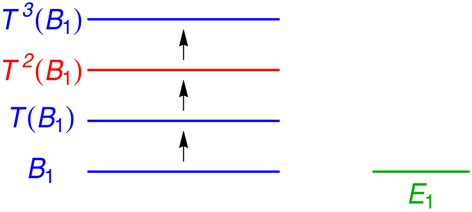


An example: Chacon map

Stage 1: This defines T on all but the top of the tower and the error set E_1.



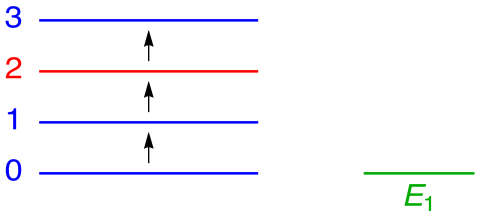
We get:

- a *tower* $\mathcal{T}_1 = \{B_1, T(B_1), ..., T^3(B_1)\}$
- which has *base* B_1 and
- *levels* B_1, $T(B_1)$, ..., $T^3(B_1)$;

- and the tower has *shape* $F_1 = \{0, 1, 2, 3\}$.

An example: Chacon map

Stage 1: This defines T on all but the top level of the tower and the error set E_1.



We get:

- a *tower* $\mathcal{T}_1 = \{T^0(B_1), T^1(B_1), T^2(B_1), T^3(B_1)\}$
- which has *base* B_1 and
- *levels* $T^0(B_1)$, $T^1(B_1)$, $T^2(B_1)$ and $T^3(B_1)$ (these levels can be labelled 0, 1, 2, and 3);
- and the tower has *shape* $F_1 = \{0, 1, 2, 3\}$.

An example: Chacon map

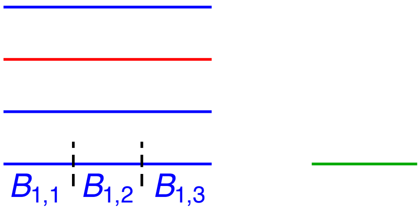
Back to Stage 0:

We can think of stage 0 as describing a tower $\mathcal{T}_0 = \{B_0\}$ of shape $F_0 = \{0\}$ and base B_0.

The stage 1 tower *refines* the stage 0 tower, i.e. each level of $\mathcal{T}_1$ is a subset of either a level (really "the" level) of $\mathcal{T}_0$ or the error set E_0.

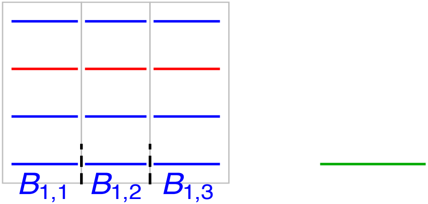
An example: Chacon map

Stage 1 → Stage 2: Cut and stack B_1, inserting spacer taken from E_1.



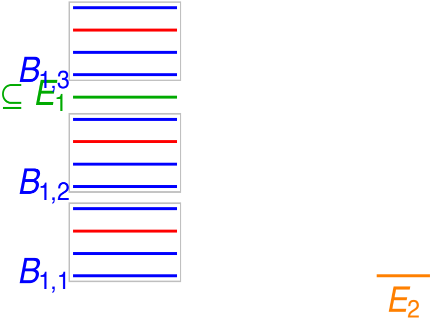
An example: Chacon map

Stage 1 → Stage 2: Cut and stack B_1, inserting spacer taken from E_1.

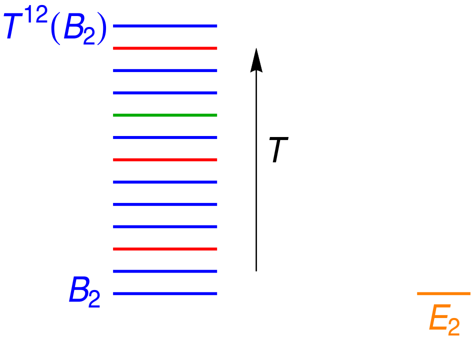


An example: Chacon map

Stage 1 $\rightarrow$ **Stage 2:** Cut and stack B_1, inserting spacer taken from E_1.



Stage 2:

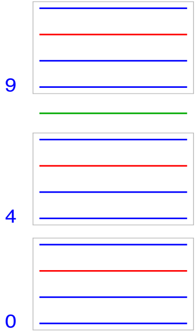


We get a tower $\mathcal{T}_2$ of shape $F_2 = \{0, ..., 12\}$, refining tower $\mathcal{T}_1$, whose base is B_2. T is now defined on more of the space than it was after Stage 1.

An example: Chacon map

Remark

Notice that levels 0, 4 and 9 of $\mathcal{T}_2$ are subsets of the base B_1 of the previous tower $\mathcal{T}_1$.



An example: Chacon map

Remark

Notice that levels 0, 4 and 9 of $\mathcal{T}_2$ are subsets of the base B_1 of the previous tower $\mathcal{T}_1$.

We will describe this by saying that the **descendants** of the stage 1 base in the stage 2 tower are the levels 0, 4 and 9, and write

$$I_{1,2} = \{0, 4, 9\}.$$

An example: Chacon map

Remark

Notice that levels 0, 4 and 9 of $\mathcal{T}_2$ are subsets of the base B_1 of the previous tower $\mathcal{T}_1$.

Similarly,

$$I_{0,2} = \{0, 1, 3, 4, 5, 7, 9, 10, 12\}$$

since those levels in the stage 2 tower are subsets of B_0, the base of the stage 0 tower.

An example: Chacon map

Stage 2 $\rightarrow$ **Stage 3** $\rightarrow$ **Stage 4** $\rightarrow \cdots$**:** Continue in this fashion to obtain a measure-preserving transformation $(X, \mathcal{X}, \mu, T)$.

In particular, the levels of the towers *generate* $\mathcal{X}$, in that any set in $\mathcal{X}$ can be approximated arbitrarily well by a union of levels taken from some tower $\mathcal{T}_n$.

For every $m \leq n$, we get a finite set $I_{m,n}$ of levels in the tower at stage n that are a subset of the base at stage m.

Defining rank one transformations

(Constructive geometric) definition of rank one (Baxter)

A **rank one $\mathbb{Z}$-action**, loosely speaking, is one that can be built via a process of cutting and stacking similar to the one used to construct the Chacon map.

Other ways to define rank one transformations:

Non-constructive geometric definition (Ornstein, Ferenczi)

Constructive symbolic definition (Kalikow, del Junco-Rudolph)

Non-constructive symbolic definition (del Junco)

generalized to group actions via (C, F)-models (Danilenko)

Adic definition (Adams-Ferenczi-Petersen)

This talk is about actions of $\mathbb{Z}^d$. We will write

$$\mathbf{T} : \mathbb{Z}^d \curvearrowright (X, \mu)$$

when $\mathbf{T}$ is an action of $\mathbb{Z}^d$ on (X, μ) by measure-preserving transformations.

The action of $\mathbf{v} \in \mathbb{Z}^d$ is denoted $\mathbf{T}^{\mathbf{v}}$.

Rank one actions of $\mathbb{Z}^d$

Definitions

A **shape** F is a nonempty, finite subset of $\mathbb{Z}^d$.

A **tower** $\mathcal{T}$ of shape F for a m.p. action $\mathbf{T} : \mathbb{Z}^d \curvearrowright (X, \mu)$ is a collection of disjoint subsets (of the same positive measure) of the form

$$\{\mathbf{T}^{\mathbf{v}}(B) : \mathbf{v} \in F\}.$$

(WLOG $\mathbf{0} \in F$, so that B is one of the levels of the tower.)

Rank one actions of $\mathbb{Z}^d$

Definitions

$\mathbf{T} : \mathbb{Z}^d \curvearrowright (X, \mathcal{X}, \mu)$ is **rank one** if there is a sequence $\{\mathcal{T}_n\}$ of towers, where $\mathcal{T}_n$ has shape F_n, so that for any $A \in \mathcal{X}$ and every n, there is a union L_n of levels of $\mathcal{T}_n$ so that

$$\lim_{n\to\infty} \mu(L_n \triangle A) = 0.$$

If each $\mathcal{T}_{n+1}$ refines $\mathcal{T}_n$, and the error sets decrease, we say $\mathbf{T}$ is **stacking rank one**.

If this definition can be satisfied so that the tower shapes $\{F_n\}$ form a Følner sequence, then we say $\mathbf{T}$ is **Følner rank one**.

Recall that $\{F_n\}$ is a Følner sequence in $\mathbb{Z}^d$ if for every $\mathbf{v} \in \mathbb{Z}^d$,

$$\lim_{n\to\infty} \frac{\#(F_n \triangle (F_n + \mathbf{v}))}{\#(F_n)} = 0.$$

Definitions

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Theorem (Robinson-Şahin 2011)

If $\mathbf{T}$ is Følner rank one, then $\mathbf{T}$ is stacking rank one (with towers of the same shapes).

"Rank one" is not "rank one"

WARNING

The definition of a rank one $\mathbb{Z}^d$-action given on the previous slide does **not** match the definition(s) of rank one given earlier for $\mathbb{Z}$-actions.

To satisfy the old definition, the shapes of the towers must be *intervals* in $\mathbb{Z}$ (shapes of the form $\{0, 1, 2, ..., h_n - 1\}$).

Here, the shapes of the towers can be *any finite subsets* of $\mathbb{Z}^d$, so thinking of this in terms of $\mathbb{Z}$-actions, our towers might look like

$$\{T^{-7}(B), T^{-4}(B), B, T^5(B), T^8(B), T^{19}(B)\},$$

etc.

“Rank one” is not “rank one”

WARNING

The definition of a rank one $\mathbb{Z}^d$-action given on the previous slide does **not** match the definition(s) of rank one given earlier for $\mathbb{Z}$-actions.

As such, this definition of rank one for $\mathbb{Z}^d$-actions generalizes not the notion of rank one for $\mathbb{Z}$-actions, but the class of $\mathbb{Z}$-actions called *funny rank one* (Thouvenot, Ferenzci).

The class of funny rank one transformations is known to be a larger class than the class of rank one $\mathbb{Z}$-actions.

Descendants

Definition

Suppose $\mathbf{T} : \mathbb{Z}^d \curvearrowright (X, \mu)$ is stacking rank one. For any $m \leq n$, set

$$I_{m,n} = \{\mathbf{i} \in F_n : \mathbf{T}^{\mathbf{i}}(B_n) \subseteq B_m\}.$$

These are the **descendants** of the base of $\mathcal{T}_m$ in the tower $\mathcal{T}_n$.

$\mathbb{Z}^d$-odometers

An important class of rank one $\mathbb{Z}^d$-actions are *odometers*:

Definition

Let $\mathfrak{G} = \{G_1, G_2, G_3, \ldots\}$ be a decreasing sequence of subgroups of $\mathbb{Z}^d$, each of which has finite index in $\mathbb{Z}^d$.

For each k, since $G_k \geq G_{k+1}$, there is a quotient map

$$q_k : \mathbb{Z}^d / G_{k+1} \to \mathbb{Z}^d / G_k.$$

Let $X_{\mathfrak{G}} = \varprojlim (\mathbb{Z}^d / G_k)$, so a point $\mathbf{x} \in X_{\mathfrak{G}}$ formally looks like

$$\mathbf{x} = (\mathbf{x}_1 + G_1, \mathbf{x}_2 + G_2, \mathbf{x}_3 + G_3, \ldots)$$

where $q_k(\mathbf{x}_{k+1} + G_{k+1}) = \mathbf{x}_k + G_k$ for all k.

$\mathbb{Z}^d$-odometers

An important class of rank one $\mathbb{Z}^d$-actions are *odometers*:

Definition (continued)

Define $\sigma_{\mathfrak{G}} : \mathbb{Z}^d \curvearrowright X_{\mathfrak{G}}$ by

$$\sigma_{\mathfrak{G}}^{\mathbf{v}}(\mathbf{x}) = (\mathbf{x}_1 + \mathbf{v} + G_1, \mathbf{x}_2 + \mathbf{v} + G_2, \ldots).$$

This action is ergodic with respect to the Haar measure on $X_{\mathfrak{G}}$.

Any action conjugate to such a $(X_{\mathfrak{G}}, \sigma_{\mathfrak{G}})$ is called a $\mathbb{Z}^d$-**odometer**.

$\mathbb{Z}^d$-odometers are Følner rank one actions (with rectangular tower shapes and empty error sets).

$\mathbb{Z}^d$-odometers

If you are used to seeing an odometer as an "add and carry" procedure on sequences of digits, here's the translation: think of

as

$$(9 + 10\mathbb{Z}, 59 + 100\mathbb{Z}, 559 + 10^3\mathbb{Z}, 4559 + 10^4\mathbb{Z}, ...) \in X_{\mathfrak{G}}$$

where $\mathfrak{G} = \{10\mathbb{Z}, 10^2\mathbb{Z}, ..., 10^k\mathbb{Z}, ...\}$.

$\mathbb{Z}^d$-odometers

Remarks

If the sequence $\mathfrak{G} = \{G_k\}$ is eventually constant, then $\#(X_{\mathfrak{G}})$ is finite, and $\sigma_{\mathfrak{G}}$ is the action of $\mathbb{Z}^d$ by translations on some finite quotient group $\mathbb{Z}^d/G_K$. We call such an action a **finite odometer** and denote it by $\tau_{\mathbb{Z}^d/G_k}$.

Otherwise, we say $\sigma_{\mathfrak{G}}$ is an **infinite odometer**. An infinite odometer is a free action if and only if $\cap_k G_k = \{\mathbf{0}\}$.

Example yielding a finite $\mathbb{Z}^2$-odometer: $G_k = 2\mathbb{Z} \times 2\mathbb{Z}\, \forall k$

Example yielding an infinite, but not free, $\mathbb{Z}^2$-odometer: $G_k = 2\mathbb{Z} \times 2^k\mathbb{Z}$

Example yielding a free $\mathbb{Z}^2$-odometer: $G_k = 2^k\mathbb{Z} \times 2^k\mathbb{Z}$

Back to the big picture

Questions

1. Which rank one $\mathbb{Z}^d$-actions factor onto *a given* $\mathbb{Z}^d$-odometer?
2. Which rank one $\mathbb{Z}^d$-actions factor onto *some* $\mathbb{Z}^d$-odometer?
3. Which rank one $\mathbb{Z}^d$-actions are conjugate to *a given* $\mathbb{Z}^d$-odometer?
4. Which rank one $\mathbb{Z}^d$-actions are conjugate to *some* $\mathbb{Z}^d$-odometer?

For $d = 1$, these questions were recently answered by Foreman, Gao, Hill, Silva and Weiss (to appear, *Isr. J. Math.*).

Note: In that setting, "rank one" means that the tower shapes must be intervals.

Back to the big picture

Questions

1. Which rank one $\mathbb{Z}^d$-actions factor onto *a given* $\mathbb{Z}^d$-odometer?
2. Which rank one $\mathbb{Z}^d$-actions factor onto *some* $\mathbb{Z}^d$-odometer?
3. Which rank one $\mathbb{Z}^d$-actions are conjugate to *a given* $\mathbb{Z}^d$-odometer?
4. Which rank one $\mathbb{Z}^d$-actions are conjugate to *some* $\mathbb{Z}^d$-odometer?

Johnson-M answered these questions for Følner rank one $\mathbb{Z}^d$-actions.

Our methods likely extend to actions of any amenable and residually finite group.

They also apply to funny rank one $\mathbb{Z}$-actions with Følner tower shapes, including certain funny rank one (but not rank one) systems studied by Ferenczi.

Results

Theorem 1 (Johnson-M)

Let $\mathbf{T} : \mathbb{Z}^d \curvearrowright (X, \mu)$ be Følner rank one and let $\sigma_{\mathfrak{G}}$ be the $\mathbb{Z}^d$-odometer coming from the sequence $\mathfrak{G} = \{G_1, G_2, G_3, ...\}$.
TFAE:

1. $(X, \mathbf{T})$ factors onto $(X_{\mathfrak{G}}, \sigma_{\mathfrak{G}})$.
2. "Eventually, most descendants are congruent mod each G_k."
   This means:
   $\forall\, k \in \mathbb{N}, \forall\, \epsilon > 0$, $\exists N \in \mathbb{N}$ so that $\forall n \geq m \geq N$, $\exists\, \mathbf{g} \in \mathbb{Z}^d$ so that
   $$\frac{\#(\{\mathbf{i} \in I_{m,n} : \mathbf{i} \not\equiv \mathbf{g} \bmod G_k\})}{\#(I_{m,n})} < \epsilon.$$

Theorem 2 (Johnson-M)

Let $\mathbf{T} : \mathbb{Z}^d \curvearrowright (X, \mu)$ be Følner rank one and let $\sigma_{\mathfrak{G}}$ be the $\mathbb{Z}^d$-odometer coming from the sequence $\mathfrak{G} = \{G_1, G_2, G_3, ...\}$. TFAE:

1. $(X, \mathbf{T})$ is conjugate to $(X_{\mathfrak{G}}, \sigma_{\mathfrak{G}})$.
2. Both a) and b) hold:
   a) "The descendants of every tower are eventually not too badly distorted mod G_k." This means: $\forall\, l \in \mathbb{N}, \forall\, \epsilon > 0$, $\exists k, N \in \mathbb{N}$ so that $\forall m \geq N$, $\exists\, D \subseteq \mathbb{Z}^d / G_k$ so that
   $$\frac{\#(\{I_{l,m} \triangle \{\mathbf{i} \in F_m : \mathbf{i} + G_k \in D\})}{\#(I_{l,m})} < \epsilon.$$
   b) Eventually, most descendants are congruent mod each G_k (same condition as Theorem 1).

Results

Theorem 3 (Johnson-M)

Let $\mathbf{T} : \mathbb{Z}^d \curvearrowright (X, \mu)$ be Følner rank one. TFAE:

1. $(X, \mathbf{T})$ is conjugate to some infinite odometer.
2. $\forall\, l \in \mathbb{N}, \forall\, \epsilon > 0$, there exists a finite index subgroup G of $\mathbb{Z}^d$ so that both a) and b) hold:
   - a) $\exists\, N_a \in \mathbb{N}$ so that $\forall\, m \geq N_a$, $\exists\, D \subseteq \mathbb{Z}^d / G$ for which
   $$\frac{\#(\{I_{l,m} \triangle \{\mathbf{i} \in F_m : \mathbf{i} + G \in D\})}{\#(I_{l,m})} < \epsilon.$$
   - b) $\forall\, \eta > 0$, $\exists\, N_b \in \mathbb{N}$ so that $\forall\, n > m \geq N_b$, $\exists\, \mathbf{g} \in \mathbb{Z}^d$ for which
   $$\frac{\#(\{\mathbf{i} \in I_{m,n} : \mathbf{i} \not\equiv \mathbf{g} \mod G\})}{\#(I_{m,n})} < \eta.$$

Some ideas in the proofs

To obtain factor maps from conditions on descendants ($2 \Rightarrow 1$)

An action $(X, \mathbf{T})$ factors onto an odometer $(X_{\mathfrak{G}}, \sigma_{\mathfrak{G}})$ if and only if it factors onto each of the finite odometers $(\mathbb{Z}^d / G_k, \tau_{\mathbb{Z}^d / G_k})$ (recall $\mathfrak{G} = \{G_1, G_2, G_3, ...\}$).

So to construct factor maps $(X, \mathbf{T}) \to (X_{\mathfrak{G}}, \sigma_{\mathfrak{G}})$, one can focus on an analysis of the "finite odometer factors" of $(X, \mathbf{T})$.

The importance of Følner tower shapes

Absent the Følner rank one assumption (even if $\mathbf{T}$ is assumed to be stacking rank one), Theorems 1-3 are false.

We use the Følner condition to show that the intersections of "enough" of the tower shapes must intersect "enough" of the cosets in each $\mathbb{Z}^d/G_k$ in "sufficiently large amounts".

This is applied in Theorem 1 to guarantee that certain proposed factor maps $(X, \mathbf{T}) \to (\mathbb{Z}^d/G_k, \tau_{\mathbb{Z}^d/G_k})$ are surjective, and used in Theorem 3 to verify that the odometer we construct from the conditions on descendants is actually infinite (and conjugate to $(X, \mathbf{T})$).

Some ideas in the proofs

To obtain conditions on descendants from factor maps ($1 \Rightarrow 2$)

If a factor map exists, then levels of the towers are "almost contained" in pullbacks of cylinder sets under the factor map.

This forces "almost containment" of most levels of later towers in pullbacks of cylinder sets.