"Equivalence" of finite and group extensions of ergodic $\mathbb{Z}^d$-actions

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Measure-preserving actions of $\mathbb{Z}^d$

Definition

A *measure-preserving (m.p.) $\mathbb{Z}^d$-action* is a quadruple $(X, \mathcal{X}, \mu, \mathbf{T})$ where $(X, \mathcal{X}, \mu)$ is a Lebesgue probability space and $\mathbf{T}$ is an action of $\mathbb{Z}^d$ on X by maps that preserve μ.

We denote the action of $\mathbf{v} \in \mathbb{Z}^d$ by $\mathbf{T_v}$.

Such an action is generated by the d commuting m.p. transformations $\mathbf{T}_{\mathbf{e}_1}, ..., \mathbf{T}_{\mathbf{e}_d}$ (where $\{\mathbf{e}_1, ..., \mathbf{e}_d\}$ is the standard basis of $\mathbb{R}^d$).

$d = 1$ corresponds to a system generated by a single measure-preserving transformation $(X, \mathcal{X}, \mu, T)$.

Cocycles and group extensions

Definition

Let $(X, \mathcal{X}, \mu, \mathbf{T})$ be a m.p. system and let G be any second countable, locally compact group. A *cocycle* for $\mathbf{T}$ is a measurable function $\sigma : X \times \mathbb{Z}^d \to G$ satisfying the following *cocycle equation*:

$$\sigma(\mathbf{T}_{\mathbf{v}}x, \mathbf{w})\sigma(x, \mathbf{v}) = \sigma(x, \mathbf{v} + \mathbf{w})$$

for a.e. $x \in X$, all $\mathbf{v}, \mathbf{w} \in \mathbb{Z}^d$.

(Throughout this talk, G is a second countable, locally compact group.)

Cocycles and group extensions

Note: When $d = 1$, a cocycle is determined by a measurable function $\sigma : X \to G$ (which we also call a cocycle), as follows:

Given cocycle σ as on the previous slide, define $\sigma : X \to G$ by

$$\sigma(x) = \sigma(x, 1).$$

Given $\sigma : X \to G$, define cocycle as on the previous slide by

$$\sigma(x, v) = \sigma(T^{v-1}x)\sigma(T^{v-2}x)\cdots\sigma(Tx)\sigma(x).$$

Cocycles and group extensions

Definition

Given $(X, \mathcal{X}, \mu, \mathbf{T})$, a *G-extension* (a.k.a. *group extension*) of $\mathbf{T}$ is a m.p. system $(X \times G, \mathcal{X} \times \mathcal{G}, \mu \times \mathit{Haar}, \mathbf{T}^\sigma)$ defined by

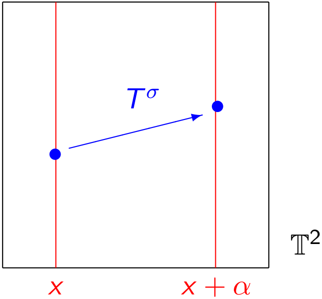
$$\mathbf{T}^\sigma_{\mathbf{v}}(x, g) = (\mathbf{T}_{\mathbf{v}} x, \sigma(x, \mathbf{v})g)$$

for all $\mathbf{v} \in \mathbb{Z}^d$, where $\sigma : X \times \mathbb{Z}^d \to G$ is a cocycle. $\mathbf{T}$ is called the *base* or *base factor* of the G-extension.

Every cocycle gives rise to a G-extension of $\mathbf{T}$, and every G-extension comes from a cocycle.

Example: skew product

Let $T : S^1 \to S^1$ be an irrational rotation by α; let $G = S^1$ and let $\sigma(x) = x$. This defines a G-extension $T^\sigma : \mathbb{T}^2 \to \mathbb{T}^2$ by $T^\sigma(x, y) = (x + \alpha, y + x)$.



Finite extensions

Notation: S_n is the symmetric group on n letters, which we will think of as acting on the finite set $[n] = \{1, 2, 3, ..., n\}$.
δ_n is uniform measure on the finite set $[n]$ (i.e. $\delta_n(x) = \frac{1}{n}$ for all x).

Definition

Let $(X, \mathcal{X}, \mu, \mathbf{T})$ be a m.p. system. A *n-point extension* of $\mathbf{T}$, a.k.a. *finite extension*, is a m.p. system $(X \times [n], \mathcal{X} \times 2^{[n]}, \mu \times \delta_n, \widetilde{\mathbf{T}}^\sigma)$ defined by

$$\widetilde{\mathbf{T}}^\sigma_{\mathbf{v}}(x, i) = (T_{\mathbf{v}} x, \sigma(x, \mathbf{v}) i)$$

where σ is a cocycle taking values in S_n.

As with group extensions, we call $\mathbf{T}$ the *base factor* of $\widetilde{\mathbf{T}}^\sigma$.

Finite extensions

Every finite extension $\widetilde{\mathbf{T}}^\sigma$ of $\mathbf{T}$ comes from a cocycle σ taking values in S_n.

$$\widetilde{\mathbf{T}}^\sigma_{\mathbf{v}}(x, i) = (\mathbf{T}_{\mathbf{v}} x, \sigma(x, \mathbf{v}) i) \qquad (i \in [n])$$

Using σ to define an S_n-extension of $\mathbf{T}$, we obtain a group extension of $\mathbf{T}$ called the *full extension* of $\widetilde{\mathbf{T}}^\sigma$.

$$\widetilde{\mathbf{T}}^\sigma_{\mathbf{v}}(x, g) = (\mathbf{T}_{\mathbf{v}} x, \sigma(x, \mathbf{v}) g) \qquad (g \in S_n)$$

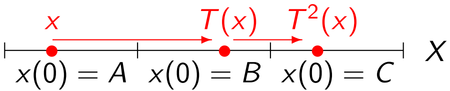
Finite extensions

Example

Let $(X, \mathcal{X}, \mu, T)$ be the full 3-shift (with alphabet A, B, C). Define $\sigma : X \to S_3$ by

$$\sigma(x) = \begin{cases} id & \text{if } x(0) = A \\ (123) & \text{if } x(0) = B \\ (132) & \text{if } x(0) = C \end{cases}$$

$x = ...ABC...$



Example

Let $(X, \mathcal{X}, \mu, T)$ be the full 3-shift (with alphabet A, B, C). Define $\sigma : X \to S_3$ by

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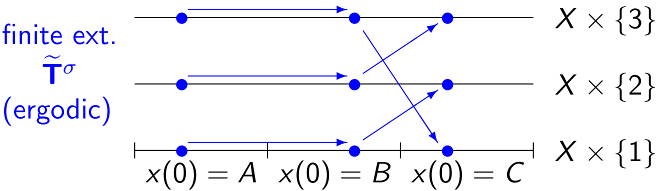

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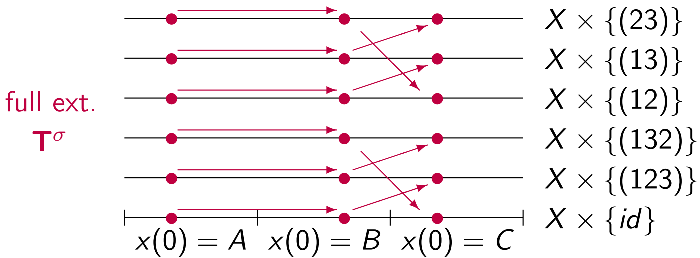

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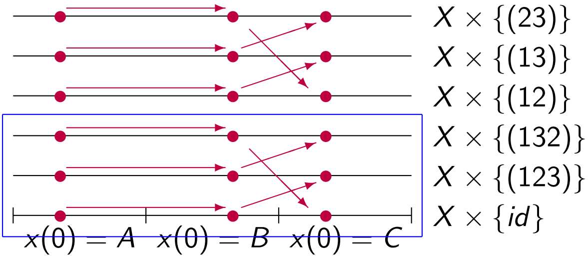
Note: $\mathbf{T}^\sigma$ is not ergodic



Relative isomorphism

Definition

Two G-extensions $\mathbf{T}^\sigma$ and $\mathbf{S}^\sigma$ (same G but not necessarily same σ) are *relatively isomorphic* if they are isomorphic via some map Φ which is measurable with respect to the base factors (i.e. given any measurable $A \subseteq Y$, $\Phi^{-1}(A \times G) = B \times G$ for some measurable $B \subseteq X$).

Every relative isomorphism Φ between two G-extensions has the form

$$\Phi(x, g) = (\phi(x), \alpha(x)g)$$

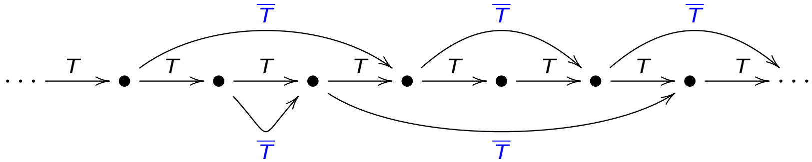
where ϕ is an isomorphism of the base factors $\mathbf{T}$ and $\mathbf{S}$, and $\alpha : X \to G$ is measurable. α is called the *transfer function* of the relative isomorphism.

(Defined similarly for finite extensions)

Speedup "equivalence"

Definition ($d = 1$)

Given m.p.t.s $(X, \mathcal{X}, \mu, T)$ and $(X, \mathcal{X}, \mu, \overline{T})$, we say $\overline{T}$ is a *speedup* of T if there exists a measurable function $v : X \to \{1, 2, 3, ...\}$ such that $\overline{T}(x) = T^{v(x)}(x)$ a.s.



Remark: by definition, speedups are (μ−a.s.) defined on the entire space, preserve μ and are $1 - 1$.

Speedup "equivalence"

Definition ($d = 1$)

Let T^σ be a G-extension of T. A *relative speedup* of T^σ is a speedup of T^σ where the speedup function v is measurable with respect to the base factor.

Definition ($d = 1$)

Let $\widetilde{T}^\sigma$ be a finite extension of T. A *relative speedup* of $\widetilde{T}^\sigma$ is a speedup of $\widetilde{T}^\sigma$ where the speedup function v is measurable with respect to the base factor.

Speedup "equivalence"

Definition ($d = 1$)

If there is a speedup of $(X, \mathcal{X}, \mu, T)$ which is isomorphic to $(Y, \mathcal{Y}, \nu, S)$, we say "you can speed up T to look like S" and write $T \rightsquigarrow S$.

Definition ($d = 1$)

If T^σ and S^σ are G-extensions, we write $T^\sigma \underset{rel}{\rightsquigarrow} S^\sigma$ if there is a relative speedup of T^σ which is relatively isomorphic to S^σ. (Similar definition for n-point extensions $\widetilde{T}^\sigma$ and $\widetilde{S}^\sigma$.)

History (speedup "equivalence" with $d = 1$)

Theorem (Arnoux, Ornstein & Weiss 1984)

If T is ergodic, and S is aperiodic, then $T \rightsquigarrow S$.

Theorem (Babichev, Burton & Fieldsteel 2013)

If T^σ (a G-extension) is ergodic and S (the base of some other G-extension) is aperiodic, then $T^\sigma \underset{rel}{\rightsquigarrow} S^\sigma$.

Theorem (Babichev, Burton & Fieldsteel 2013)

(Paraphrasing) If $\widetilde{T}^\sigma$ and $\widetilde{S}^\sigma$ are ergodic n-point extensions, then $\widetilde{T}^\sigma \underset{rel}{\rightsquigarrow} \widetilde{S}^\sigma$ if and only if $\widetilde{T}^\sigma$ has the "G_T-interchange property" and $\widetilde{S}^\sigma$ has the "G_S-interchange property", where $G_S \subseteq G_T$ (more on this later).

Speedups in $d \geq 2$

Key concept: When $d = 1$, to speed up a system means to go *forward* more quickly. What does it mean to "speed up" a system when $d \geq 2$?

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Definition

A *cone* $\mathbf{C}$ is the intersection of $\mathbb{Z}^d - \{\mathbf{0}\}$ with any open, connected subset of $\mathbb{R}^d$ bounded by d distinct hyperplanes passing through the origin.

Cones correspond to a choice of "forward" direction(s).

Speedups in $d \geq 2$

Key concept: When $d = 1$, to speed up a system means to go *forward* more quickly. What does it mean to "speed up" a system when $d \geq 2$?

Definition

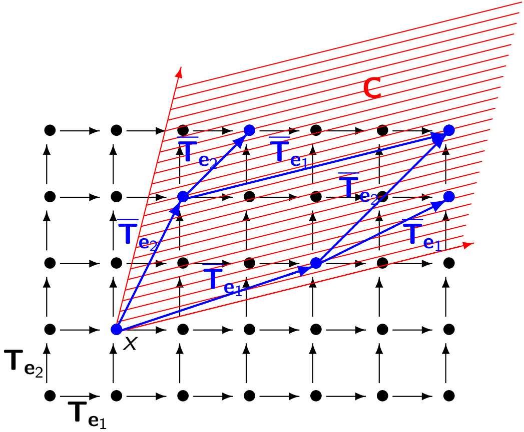
Let $\mathbf{C} \subseteq \mathbb{Z}^d$ be a cone. A $\mathbf{C}-$*speedup* of $\mathbb{Z}^d-$system $\mathbf{T}$ is another $\mathbb{Z}^d-$system $\overline{\mathbf{T}}$ (defined on the same space as $\mathbf{T}$) such that

$$\overline{\mathbf{T}}_{\mathbf{e}_j}(x) = \mathbf{T}_{\mathbf{v}_j(x)}(x)$$

for some measurable function $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_d) : X \to (\mathbf{C}^d)^d$.

Remark: The $\mathbf{v}$ must be defined so that each $\overline{\mathbf{T}}_{\mathbf{e}_i}$ and $\overline{\mathbf{T}}_{\mathbf{e}_j}$ commute (so one cannot simply speed up the $\mathbf{T}_{\mathbf{e}_j}$ independently to obtain a speedup of $\mathbf{T}$).

A picture to explain ($d = 2$)



Here, $\overline{\mathbf{T}}$ is a $\mathbf{C}$-speedup of $\mathbf{T}$. In particular, for the indicated point x, we have $\mathbf{v}(x) = ((3,1),(1,2))$.

Definition

Let $\mathbf{C} \subseteq \mathbb{Z}^d$ be any cone, and let $\mathbf{T}^\sigma$ and $\mathbf{S}^\sigma$ be G-extensions. We say $\mathbf{T}^\sigma \underset{rel}{\overset{\mathbf{C}}{\rightsquigarrow}} \mathbf{S}^\sigma$ if there is a relative $\mathbf{C}$-speedup of $\mathbf{T}^\sigma$ which is relatively isomorphic to $\mathbf{S}^\sigma$.

Theorem 1 (Johnson-M)

Let G be a locally compact, second countable group. Given any ergodic G-extension $\mathbf{T}^\sigma$ of a $\mathbb{Z}^d$-action $\mathbf{T}$ and any G-extension $\mathbf{S}^\sigma$ of an aperiodic $\mathbb{Z}^d$-action $\mathbf{S}$, and given any cone $\mathbf{C} \subseteq \mathbb{Z}^d$, $\mathbf{T}^\sigma \underset{rel}{\overset{\mathbf{C}}{\rightsquigarrow}} \mathbf{S}^\sigma$.

Sketch of proof of Theorem 1:

1. Approximate $\mathbf{S}$ by a sequence of partially-defined actions defined on larger and larger unions of Rohklin towers for $\mathbf{S}$, each union of towers being obtained from the previous one via cutting-and-stacking.
2. Choose sets in the phase space of $\mathbf{T}$ to mimic the sets found in these Rohklin towers.
3. Show that the sets from Step 2 can be realized as the phase space of a partially defined speedup of $\mathbf{T}$, with the speedup at each stage extending the one defined at the previous stage, and that these constructions can be done in a way that respects the cocycles defining $\mathbf{T}^\sigma$ and $\mathbf{S}^\sigma$.

Speedups of finite extensions

Definition

Let $\mathbf{C} \subseteq \mathbb{Z}^d$ be any cone, and let $\widetilde{\mathbf{T}}^\sigma$ and $\widetilde{\mathbf{S}}^\sigma$ be n-point extensions. We say $\widetilde{\mathbf{T}}^\sigma \underset{rel}{\overset{\mathbf{C}}{\rightsquigarrow}} \widetilde{\mathbf{S}}^\sigma$ if there is a relative $\mathbf{C}$-speedup of $\widetilde{\mathbf{T}}^\sigma$ which is relatively isomorphic to $\widetilde{\mathbf{S}}^\sigma$.

Question

Under what circumstances does $\widetilde{\mathbf{T}}^\sigma \underset{rel}{\overset{\mathbf{C}}{\rightsquigarrow}} \widetilde{\mathbf{S}}^\sigma$?

Speedups of finite extensions

Idea: Given $\widetilde{\mathbf{T}}^\sigma$ and $\widetilde{\mathbf{S}}^\sigma$, let $\mathbf{T}^\sigma$ and $\mathbf{S}^\sigma$ be the respective full extensions.

Then

$$\widetilde{\mathbf{T}}^\sigma \underset{rel}{\overset{\mathbf{C}}{\rightsquigarrow}} \widetilde{\mathbf{S}}^\sigma \Leftrightarrow \mathbf{T}^\sigma \underset{rel}{\overset{\mathbf{C}}{\rightsquigarrow}} \mathbf{S}^\sigma$$

(by using the same speedup function $\mathbf{v}$).

So if $\mathbf{T}^\sigma$ is ergodic, this is always possible by Theorem 1.

What happens if $\mathbf{T}^\sigma$ is not ergodic?

It depends on the structure of the ergodic components of $\mathbf{T}^\sigma$ and $\mathbf{S}^\sigma$. The reason is that you can make a system "less ergodic" when you speed it up, but not "more ergodic".

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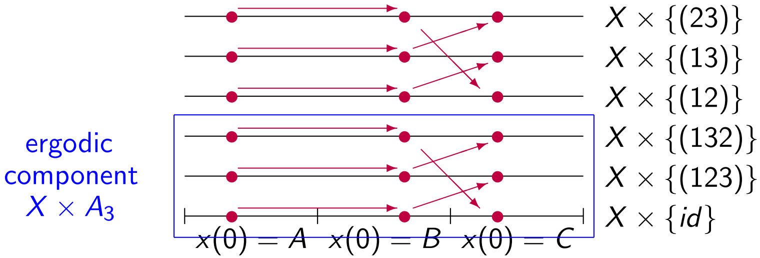
Speedups of finite extensions

Example (from before)

$$T \text{ is the full 3-shift; } \sigma(x) = \begin{cases} id & \text{if } x(0) = A \\ (123) & \text{if } x(0) = B \\ (132) & \text{if } x(0) = C \end{cases}$$

Recall that this 3-point extension was ergodic, but its full extension was not.



Bad news: In general, the full extension may not have such a simple ergodic decomposition.

Good news: Any full extension is relatively isomorphic to another S_n-extension which has $X \times G$ as one of its ergodic components, where G is some subgroup of S_n.

The set of possible Gs that can be obtained in this fashion form a conjugacy class of subgroups of S_n, and this class completely characterizes "speedupability".

Speedups of finite extensions

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Speedups of finite extensions

Lemma ($d = 1$ Gerber 1987; $d > 1$ Johnson-M)

Let $\mathbf{T}$ be an ergodic $\mathbb{Z}^d$-action and let $\widetilde{\mathbf{T}}^\sigma$ be an n-point extension of $\mathbf{T}$. Then there is a conjugacy class $gp(\widetilde{\mathbf{T}}^\sigma)$ of subgroups of S_n such that TFAE:

1. $G \in gp(\widetilde{\mathbf{T}}^\sigma)$;
2. $\widetilde{\mathbf{T}}^\sigma$ is rel. isomorphic to some other n-point extension $\widetilde{\mathbf{T}}^{\sigma'}$ of $\mathbf{T}$ such that $X \times G$ is an ergodic component of the full extension of $\widetilde{\mathbf{T}}^{\sigma'}$.

$gp(\widetilde{\mathbf{T}}^\sigma)$ is called the *interchange class* of $\widetilde{\mathbf{T}}^\sigma$.

(There is a third equivalent condition akin to what Gerber called the "G-interchange property".)

Theorem 2 ($d = 1$ Babichev, Burton & Fieldsteel 2013; $d > 1$ Johnson-M)

Let $\widetilde{\mathbf{T}}^\sigma$ and $\widetilde{\mathbf{S}}^\sigma$ be n-point extensions of ergodic $\mathbb{Z}^d$-actions $\mathbf{T}$ and $\mathbf{S}$, respectively. Then TFAE:

1. $\widetilde{\mathbf{T}}^\sigma \underset{rel}{\overset{\mathbf{C}}{\rightsquigarrow}} \widetilde{\mathbf{S}}^\sigma$;
2. For every $G_\mathbf{T} \in gp(\widetilde{\mathbf{T}}^\sigma)$, there is $G_\mathbf{S} \in gp(\widetilde{\mathbf{S}}^\sigma)$ such that $G_\mathbf{S} \subseteq G_\mathbf{T}$;
3. For some $G_\mathbf{T} \in gp(\widetilde{\mathbf{T}}^\sigma)$, there is $G_\mathbf{S} \in gp(\widetilde{\mathbf{S}}^\sigma)$ such that $G_\mathbf{S} \subseteq G_\mathbf{T}$.

Speedups of finite extensions

Idea of proof (of $3 \Rightarrow 1$**):** Suppose $G_{\mathbf{T}} \in gp(\widetilde{\mathbf{T}}^{\sigma})$; $G_{\mathbf{S}} \in gp(\widetilde{\mathbf{S}}^{\sigma})$; $G_{\mathbf{S}} \subseteq G_{\mathbf{T}}$.

WLOG the full extension of $\widetilde{\mathbf{T}}^{\sigma}$ has ergodic component $X \times G_{\mathbf{T}}$.

Construct a relative speedup on this ergodic component so that $X \times G_{\mathbf{S}}$ is an ergodic component of the speedup (easy when $d = 1$: take first return map to $X \times G_{\mathbf{S}}$; not so easy when $d > 1$).

Use Theorem 1 to speed up this speedup (restricted to its ergodic component $X \times G_{\mathbf{S}}$) to obtain a isomorphic copy of the restriction of the full extension of $\widetilde{\mathbf{S}}^{\sigma}$ to $Y \times G_{\mathbf{S}}$. Mimic this construction (performed on the full extensions) on the finite extensions to prove the result.

Relative orbit equivalence

Definition

Let $(X, \mathcal{X}, \mu, \mathbf{T})$ and $(Y, \mathcal{Y}, \nu, \mathbf{S})$ be two m.p. systems. An *orbit equivalence* is a measurable (invertible) function $\phi : X \to Y$ which preserves the measures (i.e. $\mu(\phi^{-1}(A)) = \nu(A)$ for any measurable $A \subseteq Y$) and preserves orbits (i.e. x_2 and x_1 lie on the same $\mathbf{T}$-orbit if and only if $\phi(x_1)$ and $\phi(x_2)$ lie on the same $\mathbf{S}$-orbit).

Definition

A *relative orbit equivalence* between two G-extensions (or two n-point extensions) is an orbit equivalence which is measurable with respect to the base factors.

History (orbit equivalence)

Theorem (Dye 1959)

If $d = 1$, then any two ergodic actions of $\mathbb{Z}$ are orbit equivalent.

Theorem (Connes, Feldman & Weiss 1981)

If Γ is an amenable group (this includes $\Gamma = \mathbb{Z}^d$), then any ergodic action of Γ is orbit equivalent to an ergodic action of $\mathbb{Z}$.

Theorem (Fieldsteel 1981)

If G is compact and metrizable, then any two ergodic G-extensions ($d = 1$) are relatively orbit equivalent.

Relative orbit equivalence of finite extensions

Theorem (Gerber 1987)

Let $\widetilde{T}^\sigma$ and $\widetilde{S}^\sigma$ be n-point extensions of ergodic transformations T and S, respectively. Then $\widetilde{T}^\sigma$ and $\widetilde{S}^\sigma$ are relatively orbit equivalent if and only if $gp(\widetilde{T}^\sigma) = gp(\widetilde{S}^\sigma)$.

Theorem (Johnson-M)

Let $\widetilde{\mathbf{T}}^\sigma$ be an n-point extension of ergodic $\mathbb{Z}^{d_1}$-action $\mathbf{T}$ and let $\widetilde{\mathbf{S}}^\sigma$ be an n-point extension of ergodic $\mathbb{Z}^{d_2}$-action $\mathbf{S}$. Then $\widetilde{\mathbf{T}}^\sigma$ and $\widetilde{\mathbf{S}}^\sigma$ are relatively orbit equivalent if and only if $gp(\widetilde{\mathbf{T}}^\sigma) = gp(\widetilde{\mathbf{S}}^\sigma)$.

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Relative orbit equivalence of finite extensions

The key ingredient of the proof of the ($\Leftarrow$) direction of this theorem is the following relative version of Connes-Feldman-Weiss:

Theorem (Johnson-M)

Let $\widetilde{\mathbf{T}}^\sigma$ be an n-point extension of ergodic $\mathbb{Z}^d$-action $\mathbf{T}$. Then, for any ergodic $\mathbb{Z}$-action $\widehat{T}$, there is an n-point extension $\widetilde{\widehat{T}}^\sigma$ such that:

1. $\widetilde{\mathbf{T}}^\sigma$ and $\widetilde{\widehat{T}}^\sigma$ are relatively orbit equivalent, and
2. $gp\left(\widetilde{\mathbf{T}}^\sigma\right) = gp\left(\widetilde{\widehat{T}}^\sigma\right)$.