Speedups of $\mathbb{Z}^d$-odometers

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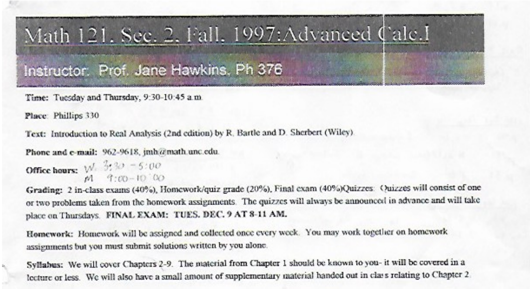
joint with Aimee S.A. Johnson (Swarthmore)

Some words about Jane



Math 121. Sec. 2. Fall. 1997:Advanced Calc.I

Instructor: Prof. Jane Hawkins, Ph 376

Time: Tuesday and Thursday, 9:30-10:45 a.m.

Place: Phillips 330

Text: Introduction to Real Analysis (2nd edition) by R. Bartle and D. Sherbert (Wiley)

Phone and e-mail: 962-9618, jmh@math.unc.edu

Office hours: W. 3:30 - 5:00
M 9:00-10:00

Grading: 2 in-class exams (40%), Homework/quiz grade (20%), Final exam (40%)Quizzes: Quizzes will consist of one or two problems taken from the homework assignments. The quizzes will always be announced in advance and will take place on Thursdays. **FINAL EXAM: TUES. DEC. 9 AT 8-11 AM.**

Homework: Homework will be assigned and collected once every week. You may work together on homework assignments but you must submit solutions written by you alone.

Syllabus: We will cover Chapters 2-9. The material from Chapter 1 should be known to you- it will be covered in a lecture or less. We will also have a small amount of supplementary material handed out in class relating to Chapter 2.

This talk is about actions of $\mathbb{Z}^d$

Definition

A $\mathbb{Z}^d-$ **measure-preserving system** ($\mathbb{Z}^d$**-m.p.s.**) is a quadruple $(X, \mathcal{X}, \mu, \mathbf{T})$ where $(X, \mathcal{X}, \mu)$ is a Lebsegue probability space and $\mathbf{T} = \{\mathbf{T_v} : \mathbf{v} \in \mathbb{Z}^d\}$ is an action of $\mathbb{Z}^d$ on X by measure-preserving transformations.

Definition

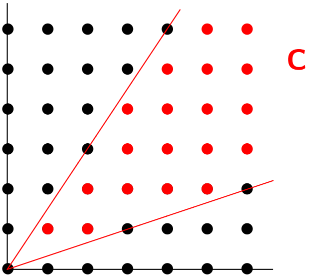
A $\mathbb{Z}^d-$ **Cantor minimal system** ($\mathbb{Z}^d$**-C.m.s.**) is a pair $(X, \mathbf{T})$ where X is a Cantor space and $\mathbf{T} = \{\mathbf{T_v} : \mathbf{v} \in \mathbb{Z}^d\}$ is a minimal action of $\mathbb{Z}^d$ on X by homeomorphisms.

In either situation, we can write $\mathbf{T} = (T_1, ..., T_d)$ where T_j is shorthand for the action of standard basis vector $\mathbf{e}_j$.

Definition

A **cone** **C** is the intersection of $\mathbb{Z}^d - \{\mathbf{0}\}$ with any open, connected subset of $\mathbb{R}^2$ bounded by d distinct hyperplanes passing through the origin.

Example in $\mathbb{Z}^2$:



Speedups of actions of $\mathbb{Z}^d$

Definition

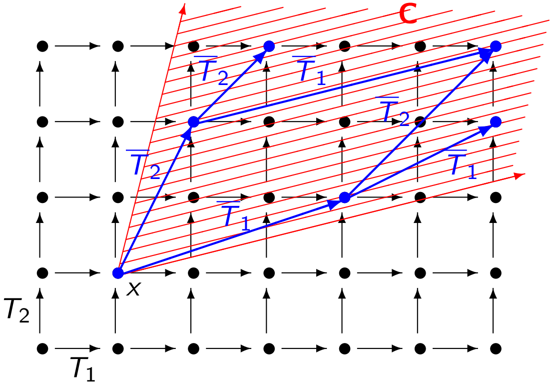
A **C−speedup** of $\mathbb{Z}^d$−action $\mathbf{T} = (T_1, ..., T_d)$ is another $\mathbb{Z}^2$−action $\mathbf{T}^{\mathbf{p}} = (\overline{T}_1, ... \overline{T}_d)$ (defined on the same space as $\mathbf{T}$) such that

$$\overline{T}_j(x) = \mathbf{T}_{\mathbf{p}_j(x)}(x)$$

for some function $\mathbf{p} = (\mathbf{p}_1, ..., \mathbf{p}_d) : X \to \mathbf{C}^d$. $\mathbf{p}$ is called the **jump function** or the **speedup function**.

Remark: The $\mathbf{p}$ must be defined so that the $\overline{T}_j$ commute (so one cannot simply speed up the generators T_j independently to obtain a speedup of $\mathbf{T}$).

A picture to explain ($d = 2$)



Here, $\mathbf{T}^{\mathbf{p}} = (\overline{T}_1, \overline{T}_2)$ is a $\mathbf{C}-$speedup of $\mathbf{T} = (T_1, T_2)$.
In particular, for the indicated point x, we have

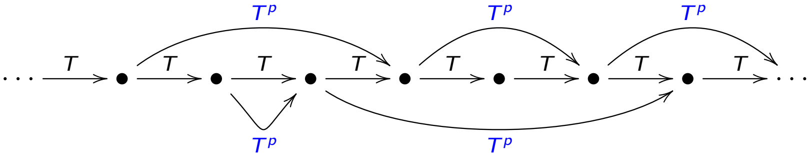
$$\mathbf{p}(x) = ((3,1),(1,2)).$$

Why is this called a "speedup"?

When $d = 1$, there are two cones:

$$\mathbf{C}_+ = \{1, 2, 3, ...\} \text{ and } \mathbf{C}_- = \{-1, -2, -3, ...\}.$$

A $\mathbf{C}_+$−speedup looks like this:



The big picture

Question

Given two $\mathbb{Z}^d$-actions $(X, \mathbf{T})$ and $(Y, \mathbf{S})$, when is there a speedup of $\mathbf{T}$ isomorphic to $\mathbf{S}$?

The word **isomorphic** means:

- measurably conjugate, if $\mathbf{T}$ and $\mathbf{S}$ are $\mathbb{Z}^d$-m.p.s.
- topologically conjugate, if $\mathbf{T}$ and $\mathbf{S}$ are $\mathbb{Z}^d$-C.m.s.

Notation

Write

$$\mathbf{T} \underset{\mathbf{C}}{\rightsquigarrow} \mathbf{S}$$

if there is a $\mathbf{C}$-speedup of $\mathbf{T}$ isomorphic to $\mathbf{S}$.

The big picture

Question

Given two $\mathbb{Z}^d$-actions $(X, \mathbf{T})$ and $(Y, \mathbf{S})$, when is there a speedup of $\mathbf{T}$ isomorphic to $\mathbf{S}$?

The word **isomorphic** means:

- measurably conjugate, if $\mathbf{T}$ and $\mathbf{S}$ are $\mathbb{Z}^d$-m.p.s.
- topologically conjugate, if $\mathbf{T}$ and $\mathbf{S}$ are $\mathbb{Z}^d$-C.m.s.

Notation

Write

$$\mathbf{T} \rightsquigarrow \mathbf{S}$$

if for any cone $\mathbf{C} \subseteq \mathbb{Z}^d$, $\mathbf{T} \underset{\mathbf{C}}{\rightsquigarrow} \mathbf{S}$.

The big picture

Question

Given two $\mathbb{Z}^d$-actions $(X, \mathbf{T})$ and $(Y, \mathbf{S})$, when is there a speedup of $\mathbf{T}$ isomorphic to $\mathbf{S}$?
The word **isomorphic** means:

- measurably conjugate, if $\mathbf{T}$ and $\mathbf{S}$ are $\mathbb{Z}^d$-m.p.s.
- topologically conjugate, if $\mathbf{T}$ and $\mathbf{S}$ are $\mathbb{Z}^d$-C.m.s.

Notation

Write

$$\mathbf{T} \overset{\text{adjective}}{\rightsquigarrow} \mathbf{S}$$

if $\mathbf{T} \rightsquigarrow \mathbf{S}$ via a speedup function $\mathbf{p}$ which is that adjective.

History of speedups: ergodic theory

Theorem (Neveu 1969)

Suppose (X, T) and (Y, S) are m.p.t.s. If $T \overset{\text{integrable}}{\rightsquigarrow} S$, then

$$h(S) = \left(\int p \, d\mu \right) h(T).$$

Theorem (Arnoux-Ornstein-Weiss 1985)

Suppose (X, T) and (Y, S) are m.p.t.s, where T is ergodic. Then $T \overset{\text{mble}}{\rightsquigarrow} S$.

Theorem (Johnson-M, 2014)

Suppose $(X, \mathbf{T})$ and $(Y, \mathbf{S})$ are $\mathbb{Z}^d$-m.p.s., where $\mathbf{T}$ is ergodic. Then $\mathbf{T} \overset{\text{mble}}{\rightsquigarrow} \mathbf{S}$.

History of speedups: ergodic theory

The basic framework of the AOW (and JM) proofs can be traced to a proof of Dye's Theorem given by Hajian, Ito and Kakutani in 1975. Recall:

Theorem (Dye 1963)

Suppose $(X, \mathbf{T})$ and $(Y, \mathbf{S})$ are ergodic $\mathbb{Z}^d$-m.p.s. Then $\mathbf{T}$ and $\mathbf{S}$ are (measurably) orbit equivalent.

Big picture idea

When $\mathbf{T}$ and $\mathbf{S}$ are orbit equivalent, we think of $\mathbf{T}$ and $\mathbf{S}$ as "having the same orbits".
When $\mathbf{T} \rightsquigarrow \mathbf{S}$, each $\mathbf{T}$-orbit is partitioned into distinct $\mathbf{S}$-orbits.

This suggests that the "speedup relation" $\rightsquigarrow$ has something to do with orbit equivalence.

History of speedups: topological dynamics, $d = 1$

Theorem (Giordano-Putnam-Skau 1995)

Let T and S be two Cantor minimal systems. Then TFAE:

1. T and S are continuously orbit equivalent.
2. T and S have isomorphic dimension groups.

Theorem (Ash)

Let T and S be two Cantor minimal systems. Then TFAE:

1. $T \overset{\text{lsc}}{\rightsquigarrow} S$.
   ("lsc" is "lower semicontinuous")
2. There is a surjective group homomorphism from the dimension group of S to the dimension group of T, preserving the positive cones and distinguished order units of those groups.

History of speedups: topological dynamics, $d = 1$

Much more restrictive things happen when one asks that the speedup function p be continuous (hence bounded, since X is compact):

Theorem (Alvin-Ash-Ormes)

Let T be an odometer (more specifically, a $\mathbb{Z}$-odometer), and suppose $T \overset{\text{cts}}{\rightsquigarrow} S$. If S is minimal, then S is a $\mathbb{Z}$-odometer which is topologically conjugate to T.

Question

What happens with continuous speedups of $\mathbb{Z}^d$-odometers?

$\mathbb{Z}^d$-odometers

$\mathbb{Z}^d$-odometers were introduced by Cortez in 2004. They are defined as follows:

The phase space

Let

$$\mathbb{Z}^d \geq G_0 \geq G_1 \geq G_2 \geq G_3 \geq \cdots$$

be a decreasing sequence of subgroups of $\mathbb{Z}^d$, each of which have finite index in $\mathbb{Z}^d$, such that $\bigcap_{j=0}^{\infty} G_j = \{\mathbf{0}\}$. Let X be the inverse limit

$$X = \varprojlim (\mathbb{Z}^d / G_j).$$

$\mathbb{Z}^d$-odometers

$\mathbb{Z}^d$-odometers were introduced by Cortez in 2004. They are defined as follows:

The phase space

Each element $\mathbf{x}$ of X is formally an infinite sequence of cosets, i.e. something like

$$\mathbf{x} = (\mathbf{x}_0 + G_0, \mathbf{x}_1 + G_1, \mathbf{x}_2 + G_2, ...)$$

where the $\mathbf{x}_j$ are "commensurate", i.e. since $G_j \geq G_{j+1}$, there's a natural map

$$\pi_j : \mathbb{Z}^d / G_{j+1} \to \mathbb{Z}^d / G_j;$$

for such a sequence to be in X we require that, for all j,

$$\pi_j(\mathbf{x}_{j+1} + G_{j+1}) = \mathbf{x}_j + G_j.$$

$\mathbb{Z}^d$-odometers

$\mathbb{Z}^d$-odometers were introduced by Cortez in 2004. They are defined as follows:

The action

X is a Cantor space, and also a topological group with addition defined coordinate-wise, where the addition in the j^{th} coordinate is the usual (vector) addition in the quotient group $\mathbb{Z}^d/G_j$.

Given any $\mathbf{v} \in \mathbb{Z}^d$, we can "convert" $\mathbf{v}$ into an element of X by setting

$$\tau(\mathbf{v}) = (\mathbf{v} + G_0, \mathbf{v} + G_1, \mathbf{v} + G_2, ...)$$

Define the action $\mathbf{T}$ of $\mathbb{Z}^d$ on X by $\mathbf{T}_\mathbf{v}(\mathbf{x}) = \mathbf{x} + \tau(\mathbf{v})$. $(X, \mathbf{T})$ is a $\mathbb{Z}^d$-C.m.s. called a **$\mathbb{Z}^d$-odometer**.

Speedups of $\mathbb{Z}^d$−odometers

Theorem (Johnson-M)

Let $\mathbf{T}$ be a $\mathbb{Z}^d$−odometer, and suppose $\mathbf{T} \overset{\text{cts}}{\rightsquigarrow} \mathbf{S}$. If $\mathbf{S}$ is minimal, then $\mathbf{S}$ is topologically conjugate to a $\mathbb{Z}^d$−odometer.

Same result as $d = 1$ (AAO), but not a similar proof.

Theorem (Johnson-M)

Let $\mathbf{T}$ be a $\mathbb{Z}^d-$odometer, and suppose $\mathbf{T} \overset{\text{cts}}{\rightsquigarrow} \mathbf{S}$. If $\mathbf{S}$ is minimal, then $\mathbf{S}$ is topologically conjugate to a $\mathbb{Z}^d-$odometer.

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Theorem (Johnson-M)

Let $\mathbf{T}$ be a $\mathbb{Z}^d-$odometer, and suppose $\mathbf{T} \overset{\text{cts}}{\rightsquigarrow} \mathbf{S}$. Even if $\mathbf{S}$ is minimal, $\mathbf{S}$ need not be topologically conjugate to $\mathbf{T}$.

Not same result as $d = 1$.

Speedups of $\mathbb{Z}^d-$odometers

Theorem (Johnson-M)

Let $\mathbf{T}$ be a $\mathbb{Z}^d-$odometer, and suppose $\mathbf{T} \overset{\text{cts}}{\rightsquigarrow} \mathbf{S}$. If $\mathbf{S}$ is minimal, then $\mathbf{S}$ is topologically conjugate to a $\mathbb{Z}^d-$odometer.

Same result as $d = 1$ (AAO), but not a similar proof.

Theorem (Johnson-M)

Let $\mathbf{T}$ be a $\mathbb{Z}^d-$odometer, and suppose $\mathbf{T} \overset{\text{cts}}{\rightsquigarrow} \mathbf{S}$. Even if $\mathbf{S}$ is minimal, $\mathbf{S}$ need not be topologically conjugate to $\mathbf{T}$.

Not same result as $d = 1$.
Actually... it is (kind of) the same result as $d = 1$, if one considers orbit equivalence.

Theorem (Johnson-M)

Let $\mathbf{T}$ and $\mathbf{S}$ be $\mathbb{Z}^d-$odometers. TFAE:

1. For some cone $\mathbf{C} \subseteq \mathbb{Z}^d$, $\mathbf{T} \underset{\mathbf{C}}{\overset{\text{cts}}{\rightsquigarrow}} \mathbf{S}$.
2. $\mathbf{T} \overset{\text{cts}}{\rightsquigarrow} \mathbf{S}$.
3. $\mathbf{T}$ and $\mathbf{S}$ are continuously orbit equivalent.

Remark: When $d = 1$, statement (3) of the above theorem implies T and S are flip conjugate (Boyle 1983, Boyle-Tomiyama 1998), and since T and S are $\mathbb{Z}$-odometers they are each isomorphic to their inverses. The AAO result follows.

Orbit equivalence and speedups of $\mathbb{Z}^d$-odometers

Theorem (Johnson-M)

Let $\mathbf{T}$ and $\mathbf{S}$ be $\mathbb{Z}^d-$odometers. TFAE:

1. For some cone $\mathbf{C} \subseteq \mathbb{Z}^d$, $\mathbf{T} \underset{\mathbf{C}}{\overset{\text{cts}}{\rightsquigarrow}} \mathbf{S}$.
2. $\mathbf{T} \overset{\text{cts}}{\rightsquigarrow} \mathbf{S}$.
3. $\mathbf{T}$ and $\mathbf{S}$ are continuously orbit equivalent.

A key concept used in the proof is that of **structural conjugacy** of odometers, recently introduced by Cortez & Medynets. This gives an algebraic condition on sequences of subgroups defining two odometers which reveals whether or not they are continuously orbit equivalent.

Orbit equivalence and speedups of $\mathbb{Z}^d$-odometers

Theorem (Johnson-M)

Let $\mathbf{T}$ and $\mathbf{S}$ be $\mathbb{Z}^d-$odometers. TFAE:

1. For some cone $\mathbf{C} \subseteq \mathbb{Z}^d$, $\mathbf{T} \underset{\mathbf{C}}{\overset{\text{cts}}{\rightsquigarrow}} \mathbf{S}$.
2. $\mathbf{T} \overset{\text{cts}}{\rightsquigarrow} \mathbf{S}$.
3. $\mathbf{T}$ and $\mathbf{S}$ are continuously orbit equivalent.

Corollary

Let $\mathbf{T}$ and $\mathbf{S}$ be $\mathbb{Z}^d-$odometers. If $\mathbf{T} \overset{\text{cts}}{\rightsquigarrow} \mathbf{S}$, then $\mathbf{S} \overset{\text{cts}}{\rightsquigarrow} \mathbf{T}$.

Orbit equivalence and speedups of $\mathbb{Z}^d$-odometers

Theorem (Johnson-M)

Let $\mathbf{T}$ and $\mathbf{S}$ be $\mathbb{Z}^d-$odometers. TFAE:

1. For some cone $\mathbf{C} \subseteq \mathbb{Z}^d$, $\mathbf{T} \overset{\text{cts}}{\underset{\mathbf{C}}{\rightsquigarrow}} \mathbf{S}$.
2. $\mathbf{T} \overset{\text{cts}}{\rightsquigarrow} \mathbf{S}$.
3. $\mathbf{T}$ and $\mathbf{S}$ are continuously orbit equivalent.

Corollary

There exists a product-type $\mathbb{Z}^d$-odometer $\mathbf{T}$ and a non-product-type $\mathbb{Z}^d$-odometer $\mathbf{S}$, such that $\mathbf{T} \overset{\text{cts}}{\rightsquigarrow} \mathbf{S}$ and $\mathbf{S} \overset{\text{cts}}{\rightsquigarrow} \mathbf{T}$.