Discontinuous identification of points by semiflows

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Spotlight on Graduate Research

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Ambrose-Kakutani Theorem

Theorem (1942) Any measure-preserving flow is measurably conjugate to a suspension flow.

For our purposes, a *measure-preserving flow*, is a system $(X, \mathcal{F}, \mu, T_t)$ where:

- X is a compact metric space
- $\mathcal{F}$ is its Borel $\sigma-$algebra
- μ is a Borel probability measure on X
- T_t is an action of $\mathbb{R}$ by invertible Borel maps that preserve μ

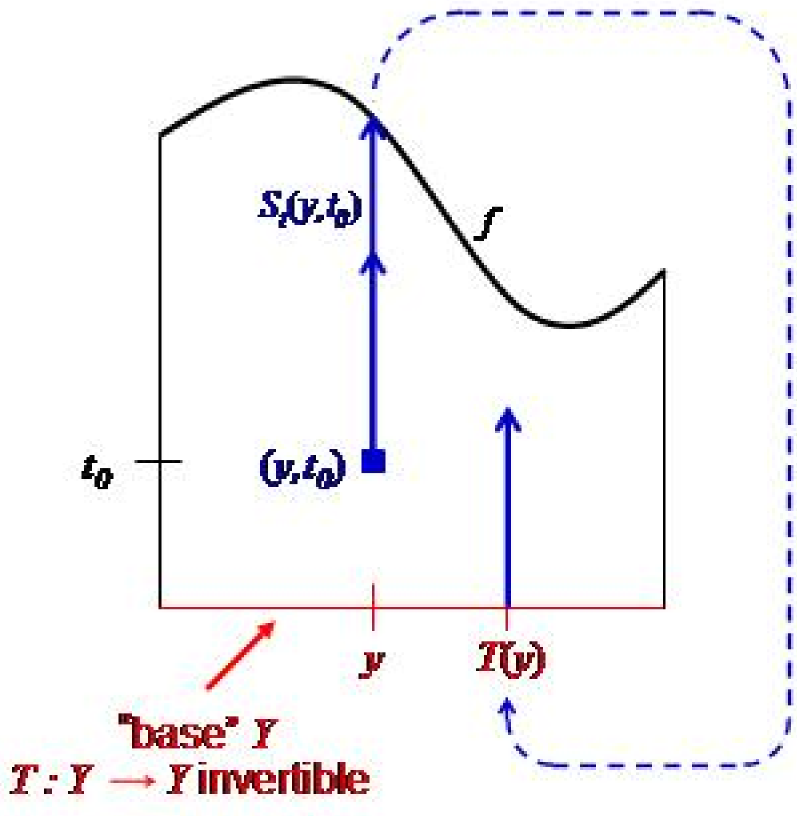
T_t is an *action* $\Leftrightarrow T_t \circ T_s = T_{t+s}$ for all t, s

T_t *preserves* $\mu \Leftrightarrow \mu(T_{-t}(A)) = \mu(A)$ for every Borel A, every t

Ambrose-Kakutani Theorem

Theorem (1942) Any measure-preserving flow is measurably conjugate to a suspension flow.

A *suspension flow*, also called a *flow under a function*, looks like the picture on the next page:



$S_t(y,t_0)$
f
t_0
(y,t_0)
y
$T(y)$
"base" Y
$T : Y \rightarrow Y$ invertible

Ambrose-Kakutani Theorem

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To say that two flows are *measurably conjugate* means that there are invariant sets of full measure in each space which can be mapped to one another by an invertible measure-preserving map α which commutes with the flows:

$$\begin{array}{ccc} X & \stackrel{\alpha}{\longrightarrow} & Y \\ \downarrow{\scriptstyle T_t} & & \downarrow{\scriptstyle S_t} \\ X & \stackrel{\alpha}{\longrightarrow} & Y \end{array}$$

(on sets of full measure in X, Y)

Ambrose-Kakutani Theorem

Theorem (1942) Any measure-preserving flow is measurably conjugate to a suspension flow.

The Ambrose-Kakutani result means that in order to study the (measure-theoretic) properties of arbitrary flows, it is sufficient to study flows under a function.

We say that flows under functions are "universal models" for flows.

Main Question

Does such a "universal model" exist for measure-preserving semiflows?

For our purposes, a *measure-preserving semiflow* is a system

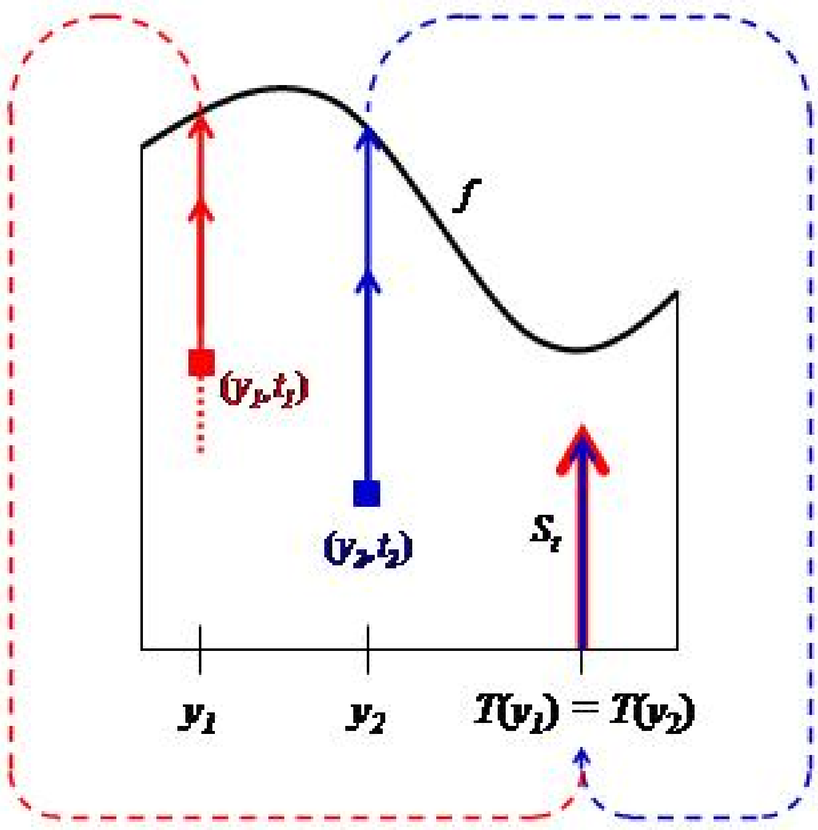
$$(X, \mathcal{F}, \mu, T_t)$$

where

- X is a compact metric space
- $\mathcal{F}$ is its Borel $\sigma-$algebra
- μ is a Borel probability measure on X
- T_t is an action of $[0, \infty)$ by (presumably non-invertible) maps that preserve μ

Candidate # 1: Suspension semiflows

If the return-time transformation in a suspension flow is not injective, then we obtain a "suspension semiflow":



Problem: Suppose the given semiflow is such that $\#(T_{-t}(x)) > 1$ for all $t > 0, x \in X$. Such a flow cannot be conjugate to a suspension semiflow because for points not at the top or bottom of the space, $\#(S_{-t}(y_1, t_1)) = 1$ for small t.

Candidate # 2: Shifts on path spaces

Suppose $X = [0,1]$ (every $(X, \mathcal{F}, \mu)$ is "the same as" $[0,1]$ with Lebesgue measure). Define for each $x \in X$ a function $f_x : [0, \infty) \to \mathbb{R}$ by

$$f_x(t) = \int_0^t T_s(x)\, ds$$

For all $x \in X$:

- $f_x(0) = 0$ and $0 \le f_x(t) \le t$
- f_x is increasing and continuous
- f_x is differentiable for Lebesgue- a.e. t

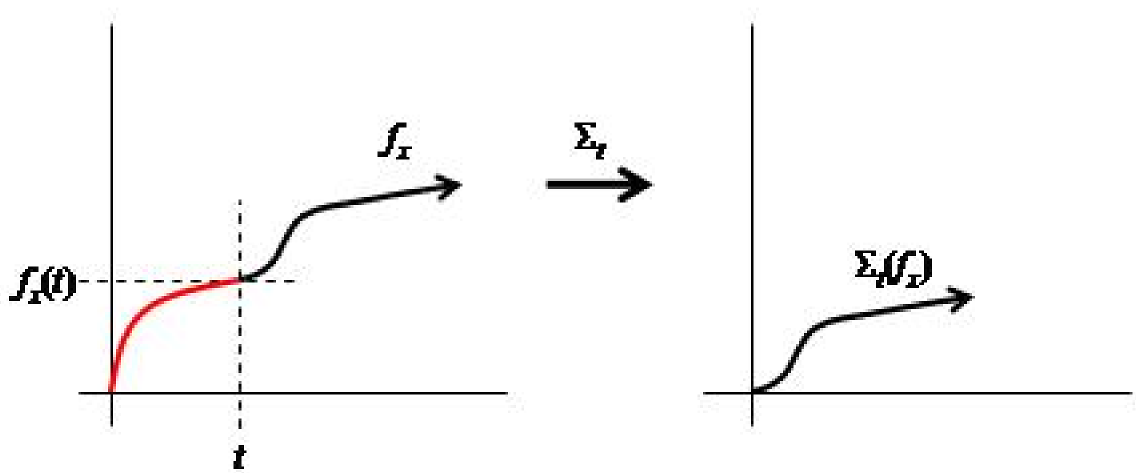
We say f_x is the "path" of x. Let Y be the set of paths coming from (X, T_t).

The shift map on Y

Given a function $f_x \in Y$, the *shift map* Σ_t is defined for each $t \geq 0$ by

$$\Sigma_t(f_x)(s) = f_x(t+s) - f_x(t).$$

Σ_t deletes the graph of f on $[0, t)$ and renormalizes so that f passes through the origin:



The shift map commutes with the semiflow: $\Sigma_t \circ (x \mapsto f_x) = (x \mapsto f_x) \circ T_t$

The problem : $x \mapsto f_x$ may not be injective

Suppose x and x' in X are distinct points such that $T_s(x) = T_s(x')$ for all $s > 0$. Then

$$f_x(t) = \int_0^t T_s(x)\, ds = \int_0^t T_s(x')\, ds = f_{x'}(t)$$

so x and x' have the same path.

In fact $f_x = f_{x'}$ iff $T_t(x) = T_t(x')\ \forall\, t > 0$.
In this case we say x and x' are *discontinuously identified* at time 0.

Discontinuous identifications are an obstacle to representing semiflows as shift maps on path spaces. We want to understand the prevalence of such behavior.

The Equivalence Classes $[x]_t$

Simplifying Assumption (*unnecessary in general*): Suppose there is a countable, dense subsemigroup S of $[0, \infty)$ such that for every $s \in S$, T_s is continuous.

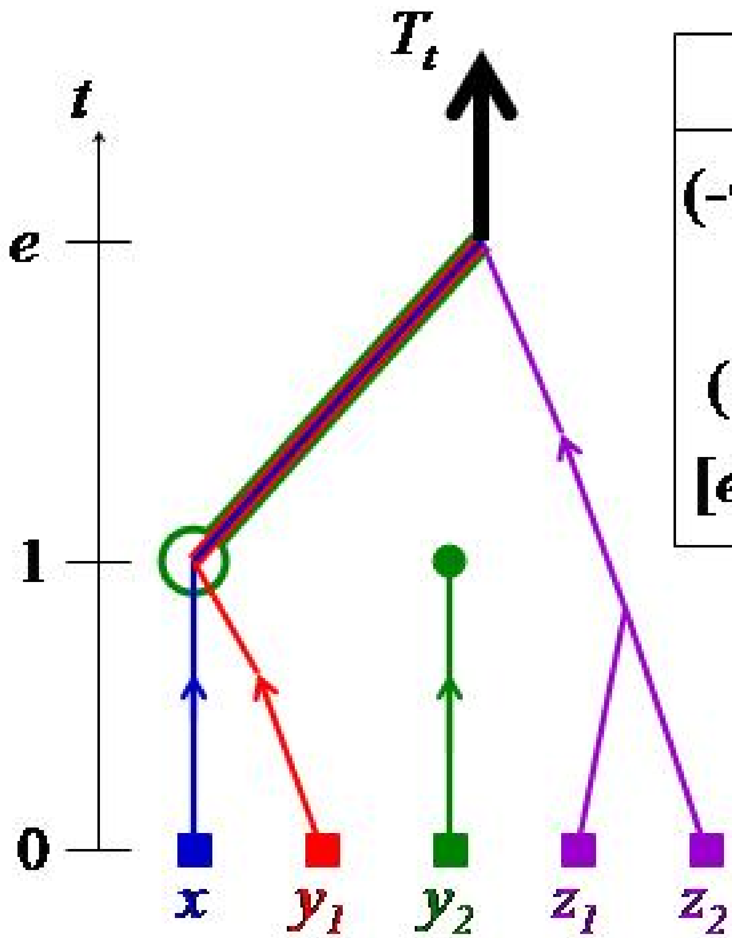
For each $x \in X$ define

$$[x]_t = \begin{cases} \bigcap_{s \geq t, s \in S} T_{-s} T_s(x) & \text{if } t \geq 0 \\ \{x\} & \text{if } t < 0 \end{cases}$$

These sets are closed and increase in t for a fixed x.

$[x]_t$ is the set of points whose forward orbits under T_t coincide with the forward orbit of x for all rational times greater than or equal to t.

An Example



t	$[x]_t$
$(-\infty, 1)$	$\{x\}$
1	$\{x, y_1\}$
$(1, e)$	$\{x, y_1, y_2\}$
$[e, \infty)$	$\{x, y_1, y_2, z_1, z_2\}$

Orbit Discontinuities

Notice $t \leq s \Rightarrow [x]_t \subseteq [x]_s$

Therefore for any $x \in X$, any $t_0 \in [0, \infty)$:

$$\overline{\bigcup_{t<t_0} [x]_t} \subseteq \bigcap_{t>t_0} [x]_t.$$

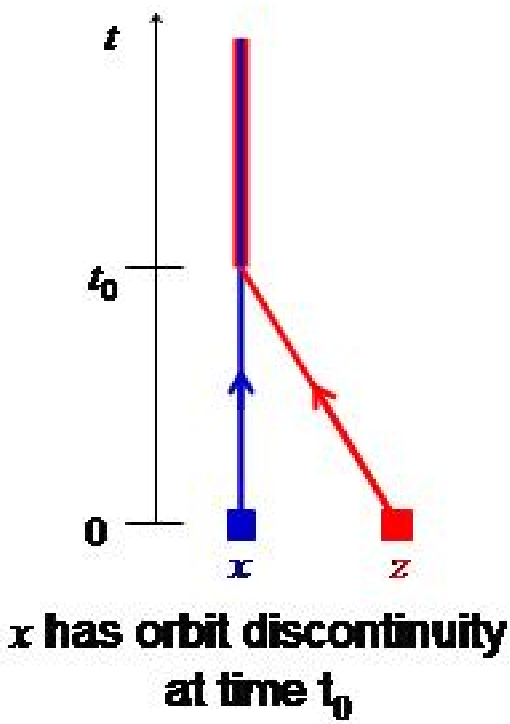
We say that $x \in X$ has an *orbit discontinuity at time t_0* if

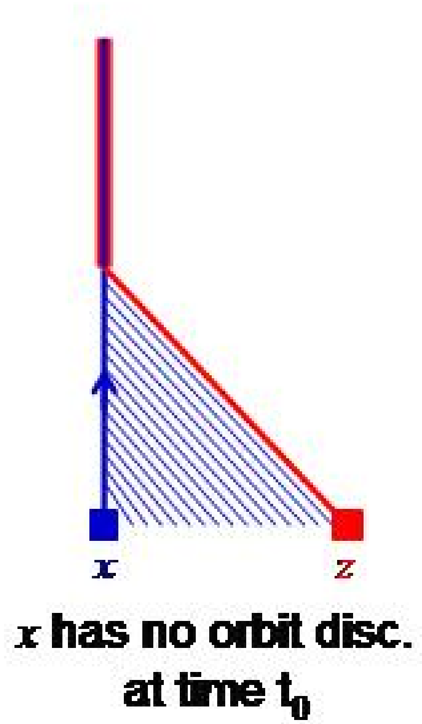
$$\overline{\bigcup_{t<t_0} [x]_t} \neq \bigcap_{t>t_0} [x]_t.$$

This is true iff there is some $z \in X$ for which:

- $T_t(z) = T_t(x)$for all $t > t_0$
- z is not the limit any sequence z_n with $T_{t_n}(z_n) = T_{t_n}(x)$ $(t_n < t_0 \, \forall n)$

Two Examples





$$\overline{\bigcup_{t<t_0} [x]_t} = \{x\}$$

$$z \in \overline{\bigcup_{t<t_0} [x]_t}$$

$$\bigcap_{t>t_0} [x]_t = \{x, z\}$$

Some results

- The set of times t where any x has an orbit discontinuity is countable.

- $x \mapsto f_x$ is not injective at $x \Leftrightarrow x$ is discontinuously identified with x' at time 0 $\Rightarrow x$ has orbit discontinuity at time 0.

- The set of points which are discontinuously identified at time 0 has measure zero with respect to any measure preserved by the semiflow.