Orbit discontinuities of Borel semiflows on Polish spaces

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Borel Semiflows

Let X be an uncountable Polish space and suppose

$$T_t : X \times \mathbb{R}^+ \to X$$

is a Borel action which preserves a Borel probability measure μ. Call (X, T_t) a *Borel semiflow*.

Question: Is there a "universal model" for such semiflows? In particular, is there one fixed Polish space $\widehat{\mathbf{X}}$ and one fixed Borel semiflow $\hat{\mathbf{T}}_{\mathbf{t}}$ on $\widehat{\mathbf{X}}$ such that every Borel semiflow is measurably conjugate to $(\widehat{\mathbf{X}}, \hat{\mathbf{T}}_{\mathbf{t}})$?

Example for discrete actions

Let Ω be a countable alphabet. Then $(\Omega^{\mathbb{Z}}, \sigma)$ is a universal model for measure-preserving $\mathbb{Z}-$ actions on a standard probability space (Sinai).

Consequence: A measure-preserving system $(X, \mathcal{F}, \mu, T)$ is determined by a shift-invariant measure on $\Omega^{\mathbb{Z}}$.

This makes it possible to describe "generic" behavior for m.p. transformations using the weak$^*-$topology on $\mathcal{M}(\Omega^{\mathbb{Z}})$.

A candidate for the universal model: shifts on path spaces

X (with topology $\mathcal{T}$) is uncountable Polish, so there is a Borel isomorphism γ between X and the Cantor set $2^{\mathbb{N}} \subset [0,1]$.

Put a topology $\mathcal{T}'$ on X so that γ is a homeomorphism; the Borel sets in the $\mathcal{T}$ and $\mathcal{T}'-$ topologies are the same. We can therefore assume X is the Cantor set.

Let Y be the set of increasing, continuous functions from $\mathbb{R}^+$ to $\mathbb{R}^+$. Y is a Polish space under the topology of uniform convergence on compact sets.

For each $x \in X$ define $f(x) \in Y$ by

$$f_x(t) = \int_0^t T_s(x)\, ds.$$

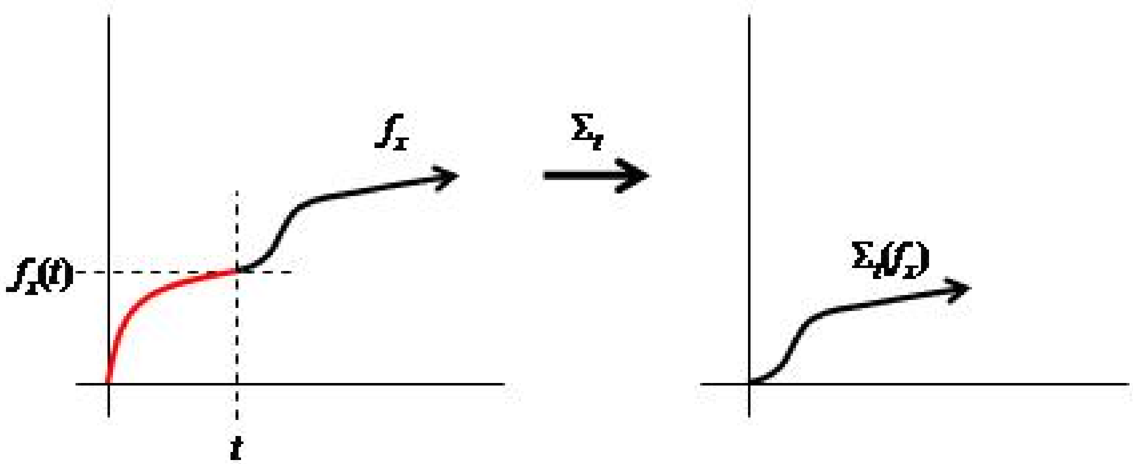
Call f_x the "path of x".

The shift map on Y

Define the *shift map* $\Sigma_t : Y \to Y$ is defined for each $t \geq 0$ by

$$\Sigma_t(f)(s) = f(t+s) - f(t).$$

Σ_t deletes the graph of f on $[0, t)$ and renormalizes so that f passes through the origin:



The shift map commutes with the semiflow:

$$\begin{array}{ccc} X & \xrightarrow{x\mapsto f_x} & Y \\ \Big\downarrow{\scriptstyle T_t} & & \Big\downarrow{\scriptstyle \Sigma_t} \\ X & \xrightarrow{x\mapsto f_x} & Y \end{array}$$

The problem : $x \mapsto f_x$ may not be injective

Suppose x and x' in X are distinct points such that $T_s(x) = T_s(x')$ for all $s > 0$. Then

$$f_x(t) = \int_0^t T_s(x)\,ds = \int_0^t T_s(x')\,ds = f_{x'}(t)$$

so x and x' have the same path.

In fact $f_x = f_{x'}$ iff $T_t(x) = T_t(x')\ \forall\, t > 0$.

We say x and x' are *instaneously discontinuously identified (IDI)* by the semiflow if $T_t(x) = T_t(x')\,\forall t > 0$.

Define $IDI(T_t) = \{x \in X : x \text{ is IDI}\}$.

Define $IDI(x) = \{t \geq 0 : T_t(x) \in \text{IDI}(T_t)\}$.

We want to understand the structure and prevalence of the IDIs of a semiflow, because IDIs are the obstacle to representing a semiflow as a shift map on a space of continuous paths.

IDIs and time-changes

Proposition: If S_t is a time change of T_t, then $IDI(S_t) = IDI(T_t)$.

Outline of Proof: To say S_t is a time change of T_t means that $\exists$ Borel cocycle

$$\alpha : X \times \mathbb{R}^+ \to \mathbb{R}^+$$

such that $S_t(x) = T_{\alpha(x,t)}(x)$.

Suppose x and y are IDI by S_t, i.e. $S_t(x) = S_t(y) \,\forall\, t > 0$.

This implies $\alpha(x,t) = \alpha(y,t) \,\forall\, t$.

So $T_t(x) = T_t(y)$ for all $t > 0$ and thus $IDI(S_t) \subseteq IDI(T_t)$.

By symmetric argument $IDI(T_t) \subseteq IDI(S_t)$.

Prevalence of IDIs

Main Theorem: For any $x \in X$, $IDI(x)$ is countable.

Consequence: Suppose that the semiflow $T_t : X \times \mathbb{R}^+ \to X$ is jointly measurable in x and t and preserves a Borel probability measure μ on X.

Then by applying the ergodic theorem, we have $\mu(IDI(T_t)) = 0$.

Outline of the Proof of the Main Theorem

Step 1: Construct an induced shift

Let S be a countable, dense, subsemigroup of $\mathbb{R}^+$ containing $\mathbb{Q}^+$.

Consider

$$\begin{aligned} X^S &= \text{set of functions } f : S \to X \\ &= \text{sequences } \{x_0, ..., x_{1/2}, ..., x_s, ...\} \text{ of} \\ &\qquad \text{points in } X \text{ indexed by } S \end{aligned}$$

X^S (with the product $\mathcal{T}'$−topology) is a Cantor space.

Define, for $s \in S$, the shift $\sigma_s : X^S \to X^S$:

$$\sigma_s(f)(t) = f(s + t).$$

σ_s maps cylinders to cylinders, so it is open, closed, and uniformly continuous.

Step 1 Continued

Define $i_T^S : X \to X^S$ by

$$i_T^S(x) = (x, ..., T_{2/5}(x), ..., T_{1/2}(x), ..., T_s(x), ...)$$

and let

$$X_1^S = \overline{i_T^S(X)}.$$

Notice that for each $s \in S$, σ_s maps X_1^S to X_1^S. In fact we have the following equivariance for $s \in S$:

$$\begin{array}{ccc} X & \xrightarrow{i_T^S} & X_1^S \\ \Big\downarrow T_s & & \Big\downarrow \sigma_s \\ X & \xrightarrow{i_T^S} & X_1^S \end{array}$$

(X_1^S, σ_s) is called an *induced shift* of (X, T_t). It models the $S-$part of the original action by continuous maps.

Step 2: Orbit discontinuities

For any $x \in X_1^S$ and any $t \in \mathbb{R}$, define

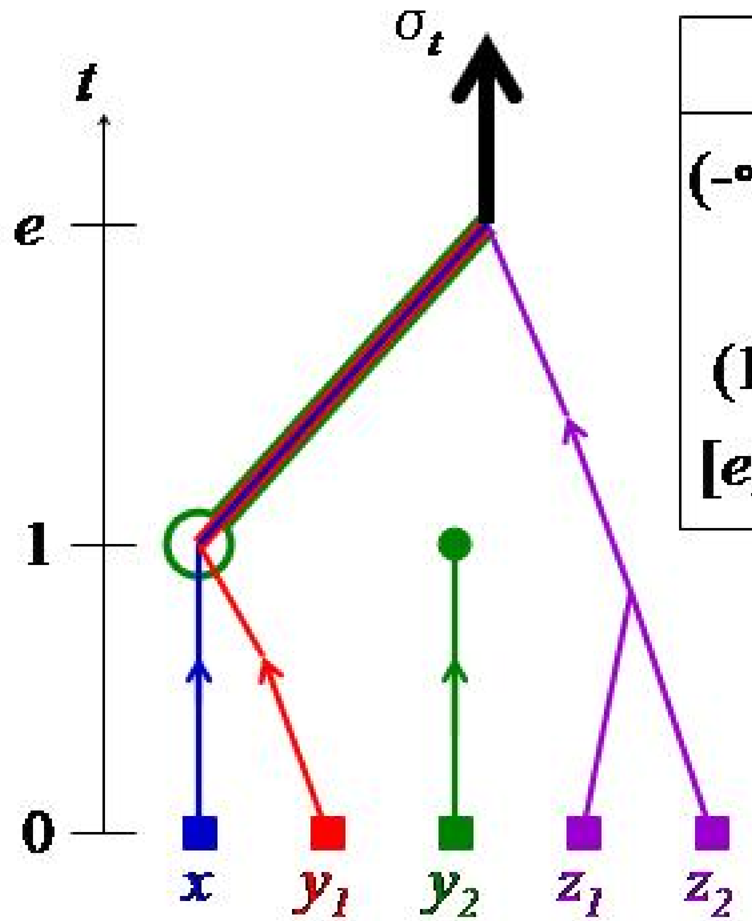
$$[x]_t = \begin{cases} \bigcap_{s \geq t, s \in S} \sigma_{-s}\sigma_s(x) & \text{if } t \geq 0 \\ \{x\} & \text{if } t < 0 \end{cases}$$

$[x]_t$ is the set of points in X_1^S which map to the same point as x under σ_s for all $s \geq t$.

For each x, $[x]_t$ is a sequence of closed sets which increase in t.

For a fixed t, $[x]_t$ partition X_1^S into closed sets.

An example of the equivalence classes $[x]_t$



t	$[x]_t$
$(-\infty, 1)$	$\{x\}$
1	$\{x, y_1\}$
$(1, e)$	$\{x, y_1, y_2\}$
$[e, \infty)$	$\{x, y_1, y_2, z_1, z_2\}$

Definition of orbit discontinuity

Recall $t \leq s \Rightarrow [x]_t \subseteq [x]_s$. Therefore $\forall\, x$ and t, we have

$$\overline{\bigcup_{t<t_0} [x]_t} \subseteq \bigcap_{t>t_0} [x]_t.$$

We say that $x \in X_1^S$ has an *$S-$orbit discontinuity at time t_0* if

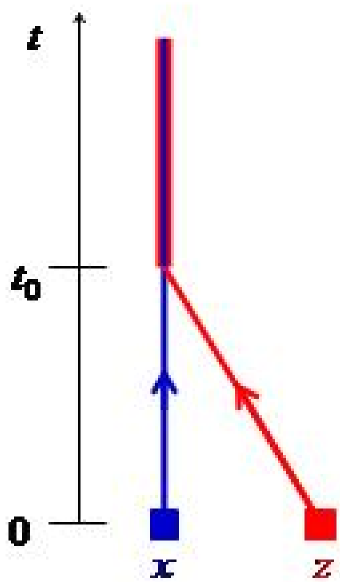
$$\overline{\bigcup_{t<t_0} [x]_t} \neq \bigcap_{t>t_0} [x]_t.$$

This is true iff there is some $z \in X_1^S$ for which:

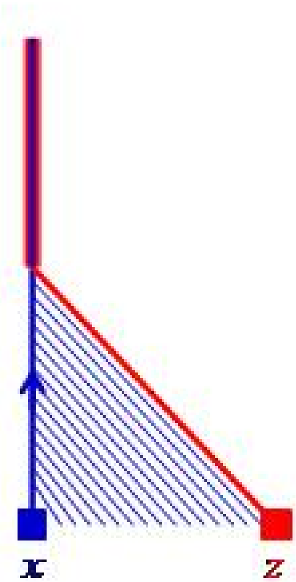
- $\sigma_s(z) = \sigma_s(x)$ for all $s \in S, s > t_0$
- z is not the limit of any sequence z_n with $\sigma_{s_n}(z_n) = \sigma_{s_n}(x)$ $(s_n < t_0\, \forall n)$

A point $x \in X$ has an *$S-$orbit discontinuity at time t_0* if $i_T^S(x) \in X_1^S$ has an $S-$orbit discontinuity at time t_0.

Two Examples



x has orbit discontinuity at time t_0



x has no orbit disc. at time t_0

$$\overline{\bigcup_{t<t_0}[x]_t} = \{x\}$$

$$z \in \overline{\bigcup_{t<t_0}[x]_t}$$

$$\bigcap_{t>t_0}[x]_t = \{x, z\}$$

Some results on orbit discontinuities

- If x has an $\mathbb{Q}^+$−orbit disc. at time t_0, then it has an S−orbit disc. at time t_0 with respect to any S containing $\mathbb{Q}^+$.

  So we say x has an *orbit discontinuity at time t_0* if it has a $\mathbb{Q}^+$−orbit discontinuity at time t_0.

  Let $D(x)$ be the set of times where x has an orbit discontinuity.

- $x \in IDI(T_t) \Rightarrow 0 \in D(x)$.

- x has an orbit discontinuity at time $t_0 \Rightarrow$ any $y \in T_{-t}(x)$ has an orbit discontinuity at time $t + t_0$.

- $IDI(x) \subseteq D(x)$.

Step 3: Show $D(x)$ is countable

Recall

$$t_0 \in D(x) \Leftrightarrow \overline{\bigcup_{t<t_0} [x]_t} \neq \bigcap_{t>t_0} [x]_t. \qquad (1)$$

Let $\mathcal{P}_k$ be a refining, generating sequence of finite partitions for X_1^S such that every atom of every $\mathcal{P}_k$ is a clopen set. Such a sequence of partitions exists for any Cantor space.

The above "non-equality" (1) is satisfied only if for some $\mathcal{P}_k$ and some atom $A \in \mathcal{P}_k$,

1. $[x]_t \bigcap A \neq \emptyset \, \forall \, t > t_0$, and

2. If B is any atom of $\mathcal{P}_k$ with $B \bigcap [x]_t \neq \emptyset$ for some $t < t_0$, then $d(a, b) > 1/k$ for any $a \in A, b \in B$.

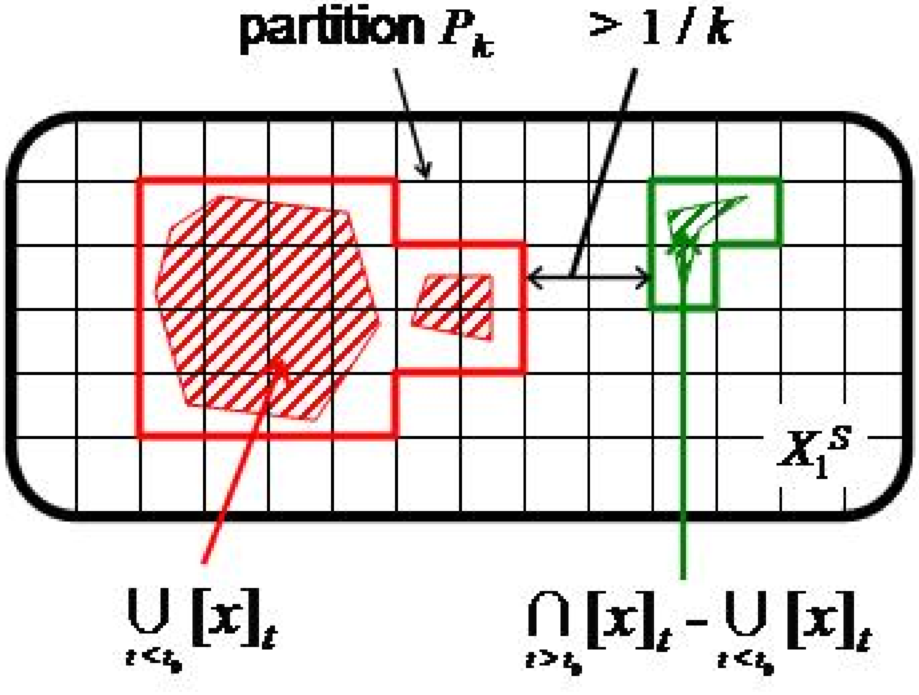
There are only countably many choices for A and k.

A picture:

Recall $t_0 \in D(x)$ only if for some k and some atom $A \in \mathcal{P}_k$,

1. $[x]_t \bigcap A \neq \emptyset \, \forall \, t > t_0$, and

2. If B is any atom of $\mathcal{P}_k$ with $B \bigcap [x]_t \neq \emptyset$ for some $t < t_0$, then $d(a, b) > 1/k$ for any $a \in A, b \in B$.



partition P_k
$> 1 / k$
X_1^S
$\bigcup_{t<t_0} [x]_t$
$\bigcap_{t>t_0} [x]_t - \bigcup_{t<t_0} [x]_t$

This is part of my Ph.D. thesis conducted under the direction of Dan Rudolph.

Preprint and slides:

http://www.math.umd.edu/$\sim$dmm