On the identification of points by Borel semiflows

David McClendon
Northwestern University

dmm@math.northwestern.edu
http://www.math.northwestern.edu/∼dmm

Universal models

We say (X, T) (where X is some set and T is some action on that set) is a *universal model* for a class of dynamical systems if every dynamical system in that class can be conjugated to (X, T).

The type of conjugacy one asks for depends on the context.

Example for discrete actions: the shift

Theorem (Sinai) *Every discrete m.p. system* $(X, \mathcal{F}, \mu, T)$ *has a countable generating partition.*

Consequence: There exists *one space* ($\Omega^{\mathbb{Z}}$) and *one action* on that space (the shift σ) such that every discrete m.p. system is measurably conjugate to $(\Omega^{\mathbb{Z}}, \sigma)$.

We say $(\Omega^{\mathbb{Z}}, \sigma)$ is a *universal model* for discrete systems.

Another way to say this is that m.p. systems "are" shift-invariant probability measures on $\Omega^{\mathbb{Z}}$.

An improvement on Ambrose-Kakutani

Theorem (Rudolph 1976) *The return-time function in the Ambrose-Kakutani picture can be chosen to take only two values* 1 *and* α *where* $\alpha \notin \mathbb{Q}$.

Consequence: m.p. flows are determined by

- a number $c \in (0, 1)$ and
- a discrete transformation (i.e. a shift-invariant measure on $\Omega^{\mathbb{Z}}$).

“Globally fixing” the path-space model

Recall: Given $x \in X$, the idea was to start with

$$f_x = \int_0^t T_s(x)\, ds$$

add “gaps” to f_x at each $t \in IDI(x)$ to obtain a new function ψ_x which is left-continuous, increasing function passing through the origin:

Distinguishing pairs

Pick refining, generating sequence of finite clopen partitions of $X_1^{\mathbb{Q}^+}$.
Suppose $x \in X$ and $t_0 \in IDI(x)$.

$j_{c_1,c_2}(x) = t$ for many pairs (c_1, c_2). Choose the coarsest partition $\mathcal{P}_k$ (smallest k) that "sees" the orbit discontinuity. In that partition, pick the collections (c_1, c_2) so that $x \in J(c_1, c_2)$ and $j_{c_1,c_2}(x) = t_0$. This pair (c_1, c_2) is called the *distinguishing pair* for the IDI.

How much gap to add?

Let β_1 be an injection of the set of pairs (c_1, c_2) into $\mathbb{N}$.

For $t_0 \in IDI(x)$, let $\beta(x, t_0) = \beta_1$(distinguishing pair for x's IDI at time t).

For fixed x, β maps $IDI(x)$ into $\mathbb{N}$ injectively.

Now add this much gap to f at time t_0:

$$2^{-\beta(x,t_0)}(T_{t_0}(x) + 2)$$

(Recall $X \subset [0, 1]$ measurably)

This adds a finite total amount of gap to f. (The total gap added is at most 3.)

Why "+ 2"?

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If one does the constructions described on the previous slide globally (for every x, t_0 with $t_0 \in IDI(x)$), we get a mapping $x \mapsto \psi_x$ where ψ_x is left-cts, increasing, and passes through the origin.

Is $x \mapsto \psi_x$ $1 - 1$?

Suppose $\psi_x = \psi_y$. Then

$$T_t(x) = (\psi_x)'(t) = (\psi_y)'(t) = T_t(y) \text{ a.s.-}t$$

so $T_t(x) = T_t(y)$ for all $t > 0$.

If $x = y$ we are done. Otherwise, x and y belong to $IDI(T_t)$.

Why "+2"? (continued)

Then $\psi_x = \psi_y$ implies the gap added at time $t_0 = 0$ to each function is the same, i.e.

$$2^{-\beta(x,0)}(T_0(x) + 2) = 2^{-\beta(y,0)}(T_0(y) + 2).$$

Rewrite this to obtain

$$2^{\beta(y,0)-\beta(x,0)} = \frac{y+2}{x+2}.$$

The left hand side is an integer power of 2; the right-hand side cannot be any integer power of 2 other than $2^0 = 1$ since both the numerator and denominator lie in $[2, 3]$. Thus $x = y$ and $x \mapsto \psi_x$ is injective.

Things are not quite right yet

We have a $1-1$ well-defined mapping $x \mapsto \psi_x$ but we have a problem:

$$\psi_{T_t(x)} \neq \Sigma_t(\psi_x)$$

This is because $\beta(x, t_0) \neq \beta(T_t(x), t_0 - t)$.

Fortunately, we can fix this.

Take a cross-section F_0 for the semiflow (not any cross-section but one with some nice properties) and "measure all β with respect to the cross-section".

That is, if $t_0 \in IDI(x)$, find where x last hits F_0 between time 0 and time t_0 (say at $T_s(x)$) and use $\beta(T_s(x), t_0 - s)$ instead of $\beta(x, t)$.

The end result

The actual amount of gap added to f_x at time t_0 is

$$2^{-\beta(T_s(x),t_0-s)}(T_t(x)+2)$$

where $T_s(x) \in F_0$ and $T_{(s,t_0]}(x) \bigcap F_0 = \emptyset$.

Theorem (M) *There exists a Polish space Y of left-continuous, increasing functions from $\mathbb{R}^+$ to $\mathbb{R}^+$ passing through the origin such that given any Borel semiflow $(X, \mathcal{F}, \mu, T_t)$, there exists a Borel injection $\Psi : X \to Y$ with*

$$\Psi \circ T_t = \Sigma_t \circ \Psi \quad \forall t \geq 0.$$

This induces a measurable conjugacy

$$(X, \mathcal{F}, \mu, T_t) \xrightarrow{\Psi} (Y, \mathcal{B}(Y), \Psi(\mu), \Sigma_t)$$