E-ergodicity and Speedups of Dynamical Systems

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A ***dynamical system*** (X, T) is a set X and a function $T: X \to X$. If x is the current state of the system, then $T(x)$ is the state of the system one unit of time later.

Examples:

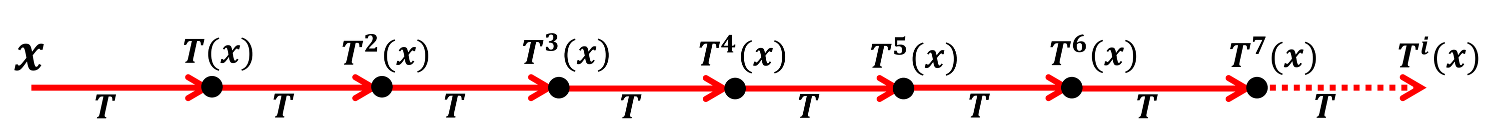
1. Suppose x is the current price of a stock. Then $T(x)$ equals tomorrow's stock price.
2. Let x be the current temperature. Then $T(x)$ is tomorrow's temperature.
3. Now let x be the number of fish in the ocean. Then $T(x)$ is the number of fish in the ocean a year from now.

Though these examples come from very different fields, the mathematics behind them reveals commonalities in their behavior. To comprehend these similarities, we need to understand the abstract properties of a mathematically defined X and T.

For the system (X, T), T^n represents

$$T^n = T \circ T \circ T \circ \ldots \circ T,$$

which consists of n compositions.



Example:
Let $T: \mathbb{R} \to \mathbb{R}$ (the real numbers) be defined as $T(x) = x^2 - x$.
Then $T(3) = 3^2 - 3 = 6$,
$T^2(3) = T(T(3)) = T(6) = 6^2 - 6 = 30$,
$T^3(3) = T(T^2(3)) = T(T(6)) = T(30) = 30^2 - 30 = 870, \ldots$

In mathematics (and its applications), we are interested in studying the average of a measurement f taken on a dynamical system.

For instance, if x represents some particle, then $f(x)$ could be the energy, the velocity , or momentum of that particle. We would be interested in the average energy in the system.

There are two ways to study this average.

(1)Time Averages:
Fix a point x in X and compute the average of f on the set $\{x, T(x), T^2(x) \ldots T^n(x)\}$

(2) Space Average
Fix a time t and compute the average value of f across X at time t.

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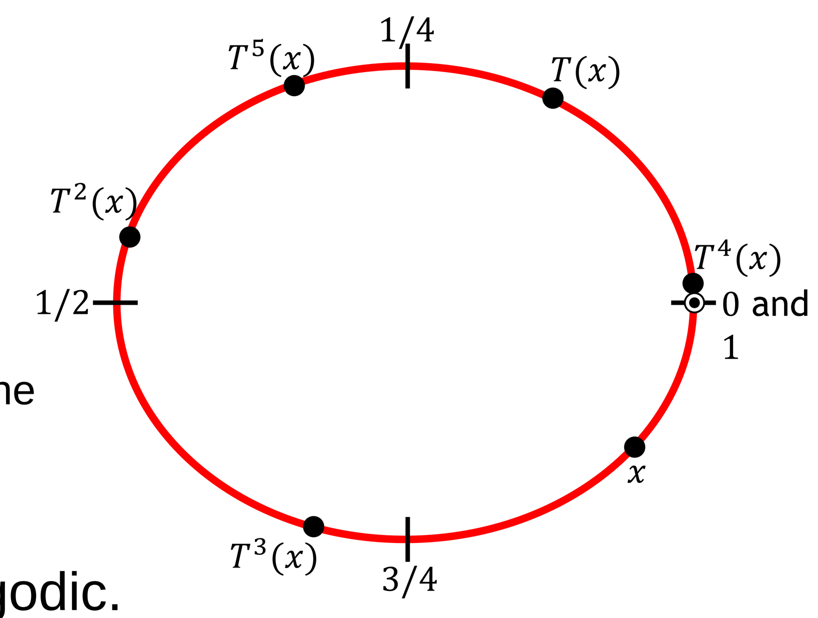
A system is called ***ergodic*** if these two averages match, i.e.

$$\lim_{n\to\infty} (\text{time average at any location}) = \text{space average at any time}.$$

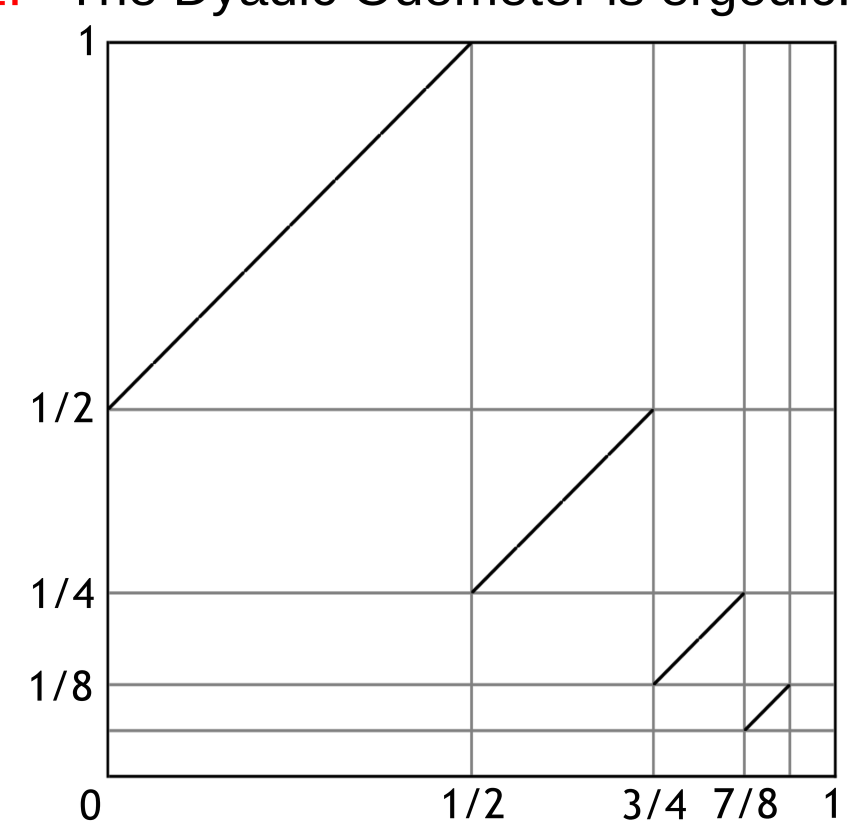
Examples:

1. A system that rotates points by an irrational angle (where angles are measured in units such that 360 degrees is one unit) is ergodic.

Over time (as n goes to infinity), eventually the entire circle will be covered uniformly and the average position will be 0. If we fix a time t and compute the average position across the circle, this is also 0.



2. The Dyadic Odometer is ergodic.



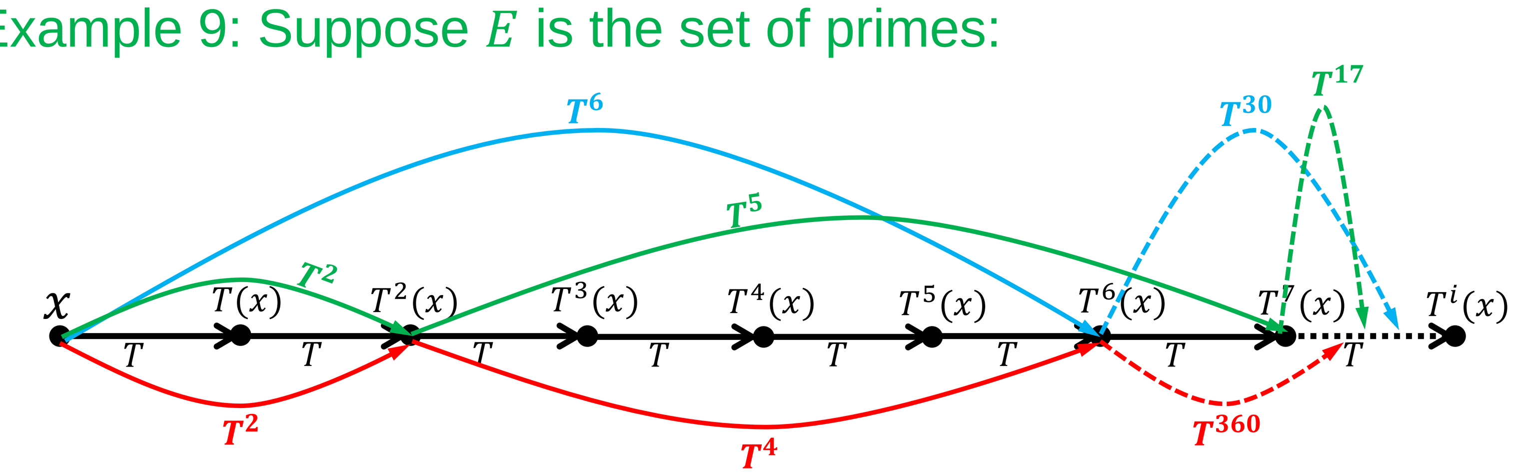
This T is ergodic, but T^2 is not ergodic because time averages for T^2 only take into account either the left or right half of [0,1].

Suppose we have the system (X, T). A ***speedup*** of T is another system $(X, \overline{T})$, where $\overline{T} = T^{p(x)}$ for some function $p: X \to \mathbb{N}$.

Definition 1.4 Given systems (X, T) and (Y, S), and given a subset E of $\mathbb{N}$, we say T can be ***E-sped up*** to S, and write $T \overset{E}{\twoheadrightarrow} S$, if there exists a speedup $\overline{T}$ of T such that

1. $\overline{T}$ looks like (is isomorphic to) S and;
2. the speedup function p of $\overline{T}$ takes values only in E.

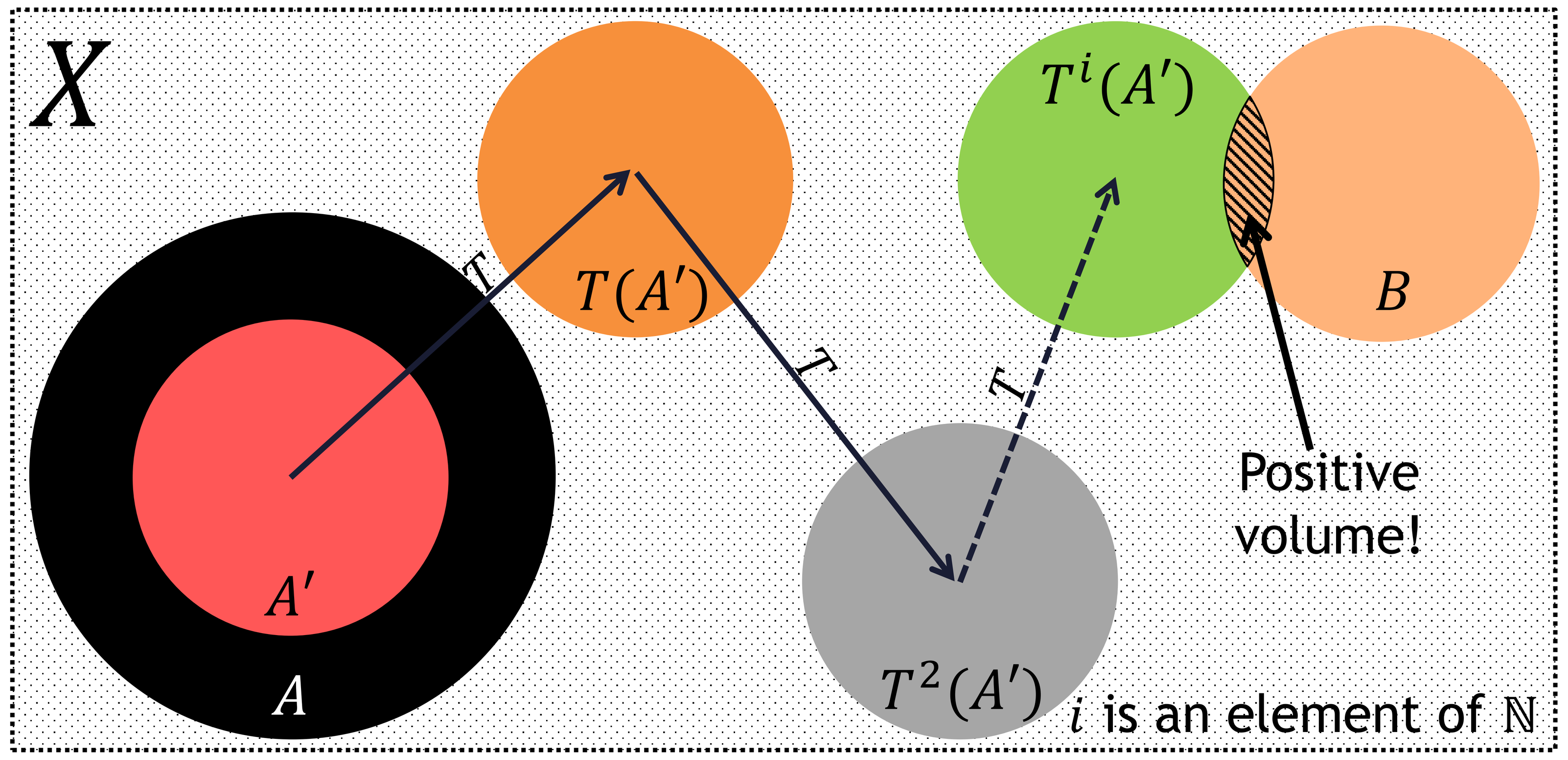
Example 7: Suppose E is the set of positive even numbers:
Example 8: Suppose E is the range of $f(x) = x^2 - x$:
Example 9: Suppose E is the set of primes:



In my undergraduate research, I discovered a new property that some dynamical systems have, I called this property ***E-ergodicity***.

Suppose we have a system (X, T) and a set E in the natural numbers. Then T is ***E-ergodic*** if for all A and B in X of positive measure, there exits an A' in A and an i in E such that the measure of A' is greater than 0 and T^i is in B.

Ergodic systems are widely studied, in particular, it is known that;
(1)If (X, T) is ergodic, then for any subsets A, B of X, there exists A' in A and i in the natural numbers such that $T^i(A') \cap B$ has positive volume.



(2) If (X, T) and (Y, S) are ergodic, then (X, T) can be sped up (non-uniformly) to obtain a mathematical copy of (Y, S).

In my research I proved the following original results:

Theorem.
Let $E \subseteq \mathbb{N}$. Also let (X, T) and (Y, S) be such that T is E-ergodic and S is $\mathbb{N}$-ergodic. Then $T \overset{E}{\twoheadrightarrow} S$.

This theorem applies in many settings, here is a list of applications:

Corollary. Let $k \in \mathbb{N}$. Also let (X, T) and (Y, S) be such that T^k is ergodic. Then for any $a \in \mathbb{N}$, $T \overset{k\mathbb{N}+a}{\twoheadrightarrow} S$.

Definition. An ***integer polynomial*** is a polynomial taking integer values on the integers.

Corollary. (X, T) is ***weak mixing*** if $(X \times X, T \times T)$ is ergodic. Also, if a system T is weak mixing, then it is ergodic. Now let (X, T) and (Y, S) be such that T is weak mixing. Then for any integer polynomial p, $T \overset{p(\mathbb{N})}{\twoheadrightarrow} S$.

Corollary. Let (X, T) and (Y, S) be such that T is totally ergodic. Then for any integer polynomial p, $T \twoheadrightarrow S$.

Corollary. Let (X, T) and (Y, S) be such that T is totally ergodic. Then for the set of primes $\mathbb{P}$, $T \overset{\mathbb{P}}{\twoheadrightarrow} S$.