Maximally continuous factors of measurable semiflows

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Actions on Polish spaces

Definition

A *Polish space* X is a separable topological space whose topology can be induced by a complete metric.
A *Polish (semi-)group* G is a set which is a (semi-)group together with a Polish topology which makes the (semi-)group actions continuous.

We think of an action of a Polish (semi-)group G on a Polish space X as a map between the Polish spaces $G \times X$ and X given by $(g, x) \mapsto gx$.

Measurable and continuous actions

Definition

An action is called *Borel measurable* if given any Borel $A \subseteq X$, the set $\{(g, x) \in G \times X : gx \in A\}$ is a Borel subset of $G \times X$.

Definition

An action is called *continuous* with respect to some topology on X if given any open $A \subseteq X$, the set $\{(g, x) \in G \times X : gx \in A\}$ is open in $G \times X$.

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Observation 1

If an action is continuous, then for each fixed $g \in G$, the map $x \mapsto gx$ is a continuous function $X \to X$.

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Observation 2

If an action is continuous, then for each fixed $x \in X$, whenever $g_n \to g$ in G, then $g_n(x) \to g(x)$ in X.

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Motivating question from descriptive set theory

What difference, if any, in set-theoretic complexity, is there between continuous actions and measurable actions?

Becker-Kechris Theorem

Theorem (Becker-Kechris, 1990)

Given a Polish space X (with topology $\mathcal{T}$) and a Borel measurable action of a Polish group G on X, there is a topology $\mathcal{T}'$ on X such that

1. $(X, \mathcal{T}')$ *is a Polish space;*
2. *The Borel sets generated by the $\mathcal{T}'-$topology coincide with the Borel sets generated by the $\mathcal{T}-$topology; and*
3. *The action of G is continuous with respect to the $\mathcal{T}'-$topology.*

We call $\mathcal{T}'$ a *nice topology* for the action.

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Equivalently, this result says that every Borel measurable group action is Borel isomorphic to a continuous one.

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In other words, one can assume that a Borel measurable group action is continuous without affecting any of the measurable structure (i.e. ergodic theory) of the action.

History

Special cases of the B-K theorem were known previously; many of these proofs incorporate ideas from dynamical systems.

- Countable groups: classical (written down by Weiss)
- $G = \mathbb{R}$: Wagh (1988) (following Ambrose-Kakutani (1941))

The proof for arbitrary Polish G by Becker and Kechris is not dynamical in nature.

Our question

Question

What happens if G is only assumed to be a Polish semigroup, rather than a group?

- Countable semi-groups: no problem (same proof as groups)
- $G = \mathbb{R}^+ = [0, \infty)$ (and other semigroups): known proofs don't work

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Fact

One reason the proof breaks down for $G = \mathbb{R}^+$ is that the Becker-Kechris result is false for actions of this semigroup.

The context

Definition

A *Borel semiflow* is a Polish space X together with a Borel measurable action of $[0, \infty)$ on X; we denote the action of $t \geq 0$ on $x \in X$ by $T_t(x)$.

Note

Given a Borel semiflow, a point $x \in X$ and a $t \geq 0$, we set $T_{-t}(x) = \{y \in X : T_t(y) = x\}$. This is a set, not a point.

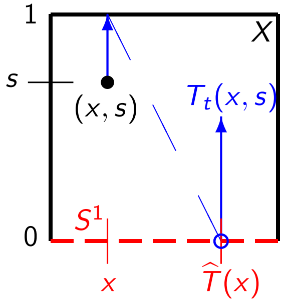
A semiflow counterexample to Becker-Kechris

Let $X = S^1 \times (0,1]$; define $\widehat{T} : S^1 \to S^1$ by $\widehat{T}(x) = 2x \mod 1$ and define

$$T_t(x,s) = \begin{cases} (x, s+t) & \text{if } s+t \leq 1 \\ (\widehat{T}(x), s+t-1) & \text{if } s+t \in (1,2] \\ (\widehat{T}^2(x), s+t-2) & \text{if } s+t \in (2,3] \\ \text{etc.} & \end{cases}$$

This semiflow takes points in X, flows them upward at unit speed, and upon reaching the "top" of X, points return to the base, flowing upward over $\widehat{T}(x)$.

Here's a picture of (X, T_t):



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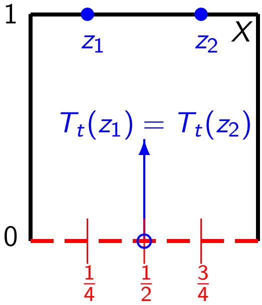
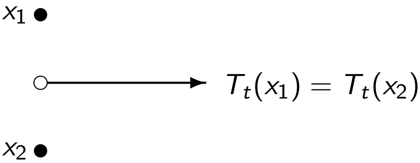


Consider the two points $z_1 = (\frac{1}{4}, 1)$ and $z_2 = (\frac{3}{4}, 1)$ in X. Notice that although $z_1 \neq z_2$, $T_t(z_1) = T_t(z_2)$ for all $t > 0$.

Instantaneous and discontinuous identifications

Definition

Given a Borel semiflow (X, T_t), we say that a point x_1 *instantaneously and discontinuously identified* (IDI) by the semiflow if there is x_2 different from x_1 such that $T_t(x_1) = T_t(x_2)$ for all $t > 0$.



Instantaneous and discontinuous identifications

Definition

We view IDIs three different ways:

1. As a subset of X:

$$IDI(T_t) = \{x \in X : x \text{ is IDI}\}$$

2. As a (Borel) equivalence relation on X:

$$\mathbf{IDI} = \{(x,y) \in X^2 : T_t(x) = T_t(y) \, \forall \, t > 0\}$$

3. For each point $x \in X$, as a collection of times:

$$IDI(x) = \{t \geq 0 : T_t(x) \in IDI(T_t)\}$$

Instantaneous and discontinuous identifications

Proposition

No semiflow with IDIs has a nice topology.

Instantaneous and discontinuous identifications

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No semiflow with IDIs has a nice topology.

Proof: Suppose a semiflow has a nice topology. Then by previous observation, for all $x \in X$,

$$\lim_{t \to 0^+} T_t(x) = T_0(x) = x,$$

in other words, x is uniquely determined by $\{T_t(x) : t > 0\}$. But points which are IDI cannot be uniquely determined by their forward orbits.

Instantaneous and discontinuous identifications

Conjecture

If a Borel semiflow has no IDIs, then it has a nice topology (i.e. a Becker-Kechris type result holds.)

Reason to believe the conjecture

Every Borel semiflow with no IDIs can be embedded in a jointly continuous action on a Polish space (M).

Revised question

Given a Borel semiflow, how close is it to a semiflow which has no IDIs?

Fixing the counterexample

Recall

$$\textbf{IDI} = \{(x,y) \in X^2 : T_t(x) = T_t(y) \text{ for all } t > 0\}$$

is a Borel equivalence relation on X. Consider the quotient $X/\textbf{IDI}$, consisting of the set of **IDI**-equivalence classes in X. This quotient can be given a Polish topology, and we obtain a factor

$$\begin{array}{ccc} X & \xrightarrow{T_t} & X \\ \downarrow \pi & & \downarrow \pi \\ X/\textbf{IDI} & \xrightarrow{T_t} & X/\textbf{IDI} \end{array}$$

of the semiflow with no IDIs.

Maximal continuous factors

Theorem (M)

Given a Borel semiflow (X, T_t), $X/\mathbf{IDI}$ *is the maximal factor of the semiflow which has no IDIs (and is therefore the maximal factor which can be embedded in a continuous action); this factor is unique up to Borel isomorphism.*

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Revised question

Given a Borel semiflow, how close is it to its factor $X/\mathbf{IDI}$?

Fixing the counterexample

In the example under consideration, the factor map $X \to X/\mathbf{IDI}$ is $1-1$ except on the points at the top of X.

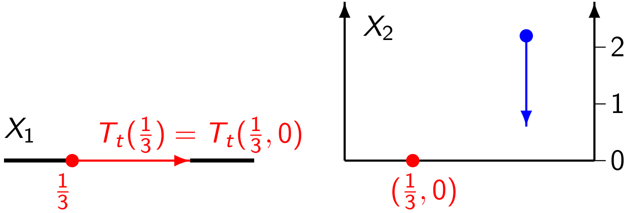
These points occur discretely along T_t-orbits, and have measure zero with respect to any Borel probability measure preserved by T_t.

So we say $X/\mathbf{IDI}$ is an *almost* $1-1$ factor of the original semiflow.

Maybe, this happens in general. Unfortunately, things don't work so nicely.

Let X be the disjoint union of $X_1 = S^1$ and $X_2 = S^1 \times [0, \infty)$. Define a semiflow on X as follows:

- Restricted to X_1, the semiflow is a rotation on X_1.
- On X_2, points flow downward at unit speed.
- A point $(x, 0)$ on the "bottom" of X_2 is IDI with the point $x \in X_1$.

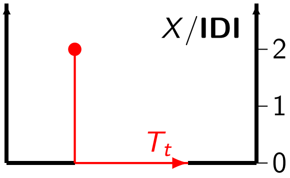


Another counterexample

If we repeat the same fix as above, we obtain

$$X/\mathbf{IDI} = S^1 \times [0, \infty)$$

and the factor semiflow on this space sends points downward at unit speed until they hit the bottom, and then rotates points on the bottom of this space.



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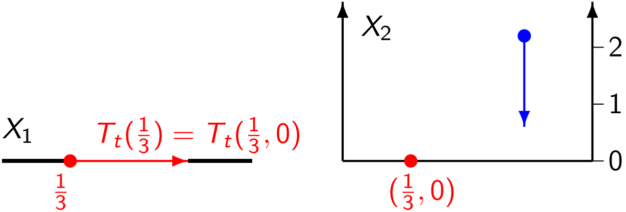

Unfortunately, the original semiflow preserves a Borel probability measure on X, namely Lebesgue measure on X_1. With respect to this measure, the factor map is $1-1$ only on a set of measure zero.

On the other hand, the original semiflow is isomorphic (in the measure-preserving sense) to the rotation on X_1, since X_1 is a (forward-)invariant set of full measure.

The general situation

It turns out that IDIs for Borel measurable semiflows which preserve a Borel probability measure, all IDIs arise in similar fashion to these counterexamples. They are either:

1. occurring countably many times along every forward orbit, and thus comprising a set of measure zero (like the first counterexample), or
2. generated by somehow "adding extra backward orbits" to the semiflow which have measure zero.

More precisely,

Main theorem

Theorem (M)

Suppose (X, T_t) is a Borel semiflow preserving a Borel probability measure μ on X (i.e. $\mu(A) = \mu(T_{-t}A)$ for all Borel A, all $t \geq 0$). As stated earlier, $X/\mathbf{IDI}$ is the maximal factor of X with no IDIs; let $\pi : X \to X/\mathbf{IDI}$ be the factor map.

Then there is a forward invariant set X_0 of full measure in X such that for $\pi^*(\mu)-$almost every $z \in X/\mathbf{IDI}$, $\pi^{-1}(z) \cap X_0$ consists of at most one point.

Restated, this result says that measure-preserving Borel semiflows have "almost 1-1" factors which have no IDIs (and can therefore be embedded inside continuous actions).

Commutative diagrams

(All maps are measurable and equivariant with T_t)

$$\begin{array}{ccc} X & \hookleftarrow & X_0 \\ \Big\downarrow \pi & & \pi|_{X_0} \Big\downarrow \text{a.s. } 1-1 \\ X/\mathbf{IDI} & & X/\mathbf{IDI} \end{array}$$

In the first counterexample:

$$\begin{array}{ccc} S^1 \times (0,1] & \overset{}{\underset{id}{\hookleftarrow}} & S^1 \times (0,1] \\ \Big\downarrow & & \Big\downarrow \\ S^1 \times [0,1) & & S^1 \times [0,1) \end{array}$$

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X/\mathbf{IDI} & & X/\mathbf{IDI}
\end{array}
$$

In the second counterexample:

$$
\begin{array}{ccc}
S^1 \uplus (S^1 \times [0,\infty)) & \hookleftarrow & S^1 \\
\Big\downarrow & & \Big\downarrow id \\
S^1 \times [0,\infty) & \hookleftarrow & S^1
\end{array}
$$

Some ideas in the proof (warning: lies)

1. Define a topology on X for which T_q is a continuous map from X to itself for every $q \in \mathbb{Q} \cap [0, \infty)$.
2. For each $x \in X$, construct a family of measures $\mu_{x,t}$ which give, for a fixed x and t, a distribution on the set $\{y \in X : T_t(y) = T_t(x)\}$.
3. Collect all the supports of all the measures $\mu_{x,t}$ and call this set X_0. Show X_0 is measurable and forward-invariant.
4. On X_0, show that every IDI arises as a "measurable" discontinuity, i.e. if $x \in X_0$ and $s \in IDI(x)$, then $x \mapsto \mu_{x,t}$ is weak$^*-$discontinuous in t at $t = s$.
5. Show that the "measurable" discontinuities are a set of measure zero (in fact, they occur countably often along forward orbits (M, 2009)).

Further directions for study

1. Resolving the conjecture stated earlier (does the absence of IDIs ensure a nice topology for semiflows?)
2. Other semigroups
3. Appropriate equivalence theory for semiflows and endomorphisms
    - (measure-theoretic) isomorphism theory for "Bernoulli" semiflows
    - classification of Borel semiflows up to Borel time-change (measure-preserving case: Lin-Rudolph '04)
    - classification of Borel endomorphisms up to "descriptive Kakutani equivalence" (begun by Miller-Rosendal)