Dynamics of the family $\lambda(z + \frac{1}{z} + 1)$

Anzhané Lance

Ferris State University
Big Rapids, MI, USA

July 14, 2017

What is a dynamical system?

Definition

A **dynamical system** is a pair (X, f) where X is a set and $f : X \to X$.

Think of $x \in X$ as being the current "state" of the system. Then $f(x)$ represents the state of the system tomorrow, and $f^2(x) = f(f(x))$ represents the state two days from now, etc. More generally, we define

$$f^n(x) = f(f(f(f...(f(x))));$$

the functions f^n are called **iterates** of f.

One of the main goals of dynamical systems is to explain/predict what will happen as $n \to \infty$.

Rational functions and the Riemann sphere

We study dynamical systems where the phase space X is the **Riemann sphere** $\mathbb{C}_\infty$.

To visualize $\mathbb{C}_\infty$, take the complex plane $\mathbb{C}$ and imagine "wrapping" the plane around a sphere. This plane will cover all but the north pole of the sphere, and we "fill in" the north pole by adding a point called ∞. Therefore

$$\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}.$$

Thus we think of a complex number as a point on a sphere, where 0 is the bottom of the sphere, ∞ is the top of the sphere and the equator consists of the complex numbers whose absolute value is 1.

Rational functions and the Riemann sphere

The functions we study are rational functions:

Definition

A **rational function** is a function f from $\mathbb{C}_\infty$ to itself of the form $f(z) = \frac{g(z)}{h(z)}$ in which $g(z)$ and $h(z)$ are polynomials with no common zeros.

This is a natural class of functions to study, because rational functions are the only meromorphic functions on $\mathbb{C}_\infty$.

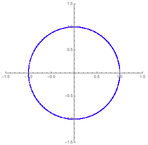
History of complex dynamics

The dynamics of complex rational maps was first explored by French mathematicians Fatou and Julia in the 1910s. They showed that for any rational function f, we can divide the Riemann sphere into two sets: the **Fatou set** $F(f)$, i.e. the set of points which behave predictably under iteration, and its complement the **Julia set** $J(f)$, i.e. the set of points that behave chaotically under iteration.

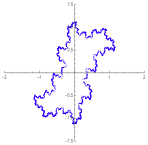
Identifying and studying these two sets is vital to understanding the dynamics of complex rational maps.

History of complex dynamics

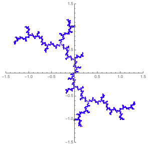
In the 1970s, Benoit Mandelbrot reignited interest in complex dynamics by using computer calculations to show the highly intricate structure of the Fatou and Julia sets (and other associated sets). Here are some examples of Julia sets of maps of the form $f(z) = z^2 + c$:



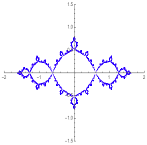
$J(z^2)$



$J(z^2 + .1 - .6i)$



$J(z^2 + i)$



$J(z^2 - 1)$

Since then, the field has been a highly active area of mathematics research.

Definitions

Let $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a rational function.

Definition

A **fixed point** of the system is a point $z \in \mathbb{C}_\infty$ such that $f(z) = z$.

Definition

A **periodic point** of the system is a fixed point of f^k for some $k \in \mathbb{N}$. If z is a periodic point, then the smallest $k > 0$ such that $f^k(z) = z$ is called the **period** of z.

Definition

Let z be a periodic point of f of period k. Then:

- z is called **attracting** if $|(f^k)'(z)| < 1$.
- z is called **repelling** if $|(f^k)'(z)| > 1$.

Definition

Let $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a rational map. Then the **Julia set** $J(f)$ is the closure of the set of repelling periodic points of f. The complement of the Julia set is the **Fatou set** $F(f)$.

(There are lots of other equivalent definitions of $F(f)$ and $J(f)$.)

All quadratic rational functions are conjugate to one of two forms:

$$f_\lambda(z) = \lambda\left(z + \frac{1}{z} + \beta\right) \quad \text{or} \quad g_c(z) = z^2 + c.$$

The family $g_z(c) = z^2 + c$ has been studied completely (it leads to the well-known Mandelbrot set), whereas $f_\lambda(z) = \lambda\left(z + \frac{1}{z} + \beta\right)$ has only been studied in detail in two special cases ($\beta = 0$ or 2) by Hawkins and McClendon.

We study the family of quadratic rational maps

$$\left\{ f_\lambda(z) = \lambda\left(z + \frac{1}{z} + 1\right) : \lambda \in \mathbb{C} - \{0\} \right\}.$$

λ is called the **parameter** of this family.

Main questions

1. Divide the set of parameters λ into regions where the dynamical behavior of f_λ depends only on the region that contains λ and not on the exact value of λ.
2. For λ in each of these regions, determine the topological characteristics of $J(f_\lambda)$ and analyze the long-term behavior under iteration of $\lambda(z + \frac{1}{z} + 1)$.
3. Compare and contrast the answers to questions 1 and 2 to similar questions asked about other families of quadratic rational maps ($z^2 + c$ and $\lambda(z + \frac{1}{z} + 2)$ especially).

Fixed and periodic points

First, for our family $f_\lambda(z) = \lambda(z + \frac{1}{z} + 1)$:

1. f_λ has two fixed points:

$$z_+ = \frac{-\lambda + \sqrt{-3\lambda^2 + 4\lambda}}{2\lambda - 2}; z_- = \frac{-\lambda - \sqrt{-3\lambda^2 + 4\lambda}}{2\lambda - 2}.$$

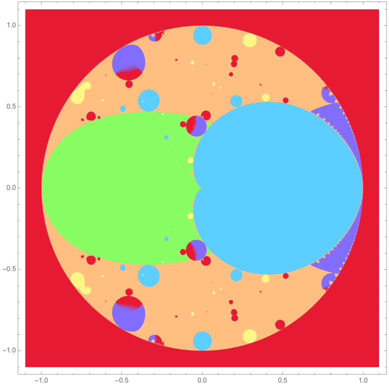
2. f_λ has one period 2-cycle $\{p_+, p_-\}$:

$$p_{+,-} = \frac{-(\lambda + 1) \pm \sqrt{1 - 3\lambda^2 - 2\lambda}}{2\lambda + 2}$$

Decomposition of the parameter space

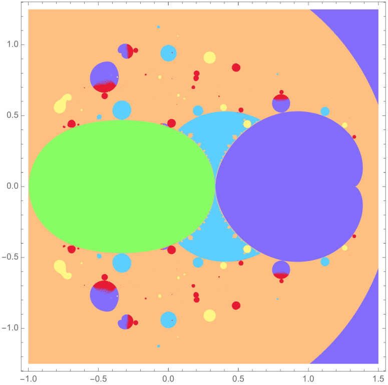
We divide the parameter space into disjoint regions by studying what happens to the critical points of f_λ (namely ± 1) under iteration.

First, the critical point $+1$:



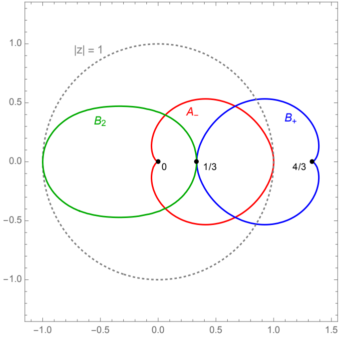
Decomposition of the parameter space

Next, a picture of what happens to -1 under iteration:



Decomposition of the parameter space

Superimposing these pictures (and simplifying) leads to the following subsets of the parameter space:



These regions indicate the values λ for which f_λ has an attracting fixed point and/or attracting period 2 cycle.

Julia sets

Here are some Julia sets $J(f_\lambda)$ for various λ:

Figure 5

Julia_point5_pluspoint2i.pdf

$\lambda = 0.5473 + 0.2367i$
$\lambda \in A_- \cap B_+$

Julia_point2_pluspoint1.pd

Julia set of $(0.223 + 0.1225i)(z + \frac{1}{z} + 1)$. The fixed points are indicated in red; z_+ is attracting; z_- is repelling; the attracting 2-cycle is shown in green

Julia sets

Here are some Julia sets $J(f_\lambda)$ for various λ:

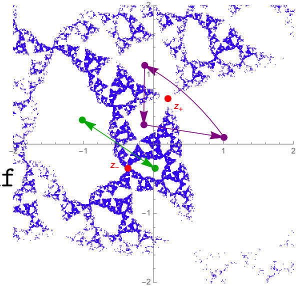
Figure 6

Julia_point89_pluspoint36.pdf

..

Julia set of $(0.8913 + 0.3618i)(z + \frac{1}{z} + 1)$. The fixed points are indicated in red; z_+ is attracting; z_- is repelling.

Figure 7



Julia set of $(-0.04452+0.3812i)(z+\frac{1}{z}+1)$. The repelling fixed points are indicated in red; the attracting 2-cycle is shown in green; the attracting 3-cycle is shown in

Future goals

In this talk, we have focused on the behavior of the family $\lambda(z + \frac{1}{z} + 1)$ where λ is such that the map has an attracting fixed point and/or attracting 2-cycle. We are currently studying the behavior when the attracting cycles have larger period.