What do orbits look like?

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Dynamical systems

A *dynamical system* is anything (quantifiable) that changes with the passage of time.

Examples of "real-world" dynamical systems:

- The temperature
- The price of a stock
- The spin of an electron
- The rabbit population in Pennsylvania
- The velocity of flowing water

Mathematical formulation

To formulate a dynamical system mathematically, we need to specify two things:

Mathematical formulation

To formulate a dynamical system mathematically, we need to specify two things:

1. The phase space

The *phase space* X of a dynamical system is the set of all possible "positions" of the system.

For example, if the system is keeping track of the price of a stock, X is the set of all possible stock prices.

Mathematical formulation

To formulate a dynamical system mathematically, we need to specify two things:

2. The evolution rule

The *evolution rule* T^t of a dynamical system is the formula that tells you, given your current position x and any amount of time t, your position at time t (as a function of x and t).

For example, if the system is keeping track of a stock price, if the current price is 30, then $T^{12}(30)$ is the price of the stock in 12 days.

Mathematical formulation

To formulate a dynamical system mathematically, we need to specify two things:

2. The evolution rule

The evolution rule for a dynamical system has to obey some laws:

1. For each time $t \geq 0$, T^t is a function from X to X.
2. T^0 is the identity map ($T^0(x) = x$ for all $x \in X$)
3. $T^{s+t}(x) = T^s(T^t(x))$ for all times $s, t \geq 0$ and all $x \in X$.

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2. The evolution rule

Since a primary goal of dynamical systems is to "predict the future", i.e. say something about $T^t(x)$ for large values of t, we will assume all maps T^t are surjective (otherwise, make X smaller).

Mathematical formulation

To formulate a dynamical system mathematically, we need to specify two things:

2. The evolution rule

If the functions T^t are all invertible (a.k.a. one-to-one, injective), we call the dynamical system *invertible*; in this situation we see that T^{-t} is a function which is the inverse of T^t for all t.

To formulate a dynamical system mathematically, we need to specify two things:

Definition

A *dynamical system* is be a pair (X, T^t) where X is some set and T^t is some collection of functions satisfying the laws described here.

The nature of time

Loosely speaking, dynamical systems split into two types depending on whether the set of times t under consideration is *discrete* or *continuous*.

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Discrete-time dynamical systems

Here we only allow values of t that are integers, i.e. there is...

$$\text{time } t = 0 \leftrightarrow \text{ the present}$$

$$\text{time } t = 1 \leftrightarrow \text{ one unit of time from now}$$

$$\text{time } t = -6 \leftrightarrow \text{ six units of time ago}$$

but no time $t = \sqrt{2}$ or $\frac{3}{4}$ or π, etc.

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Discrete-time dynamical systems

In this situation, the function $T^1 : X \to X$ determines the entire dynamical system because

$$T^2(x) = T^{1+1}(x) = T^1(T^1(x)) = (T^1 \circ T^1)(x)$$

and more generally,

$$T^t(x) = T^{1+1+\ldots+1}(x) = T^1(T^1(\cdots(T^1(x)))) = (T^1 \circ \ldots \circ T^1)(x).$$

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Discrete-time dynamical systems

In a discrete-time dynamical system, the future dynamics can be represented by the following diagram:

$$x = T^0(x) \rightarrow T^1(x) \rightarrow T^2(x) \rightarrow \ldots \rightarrow T^t(x) \rightarrow T^{t+1}(x) \rightarrow \ldots$$

Two examples of discrete-time systems

Example 1

Let $X = \mathbb{R}$ and let $T^1(x) = -x$.
Then $T^2(x) = T^1(T^1(x)) = -(-x) = x$, and similarly

$$T^t(x) = \begin{cases} x & \text{if } t \text{ is even} \\ -x & \text{if } t \text{ is odd} \end{cases}$$

Two examples of discrete-time systems

Example 1

What's more, in this example if we know that our current position is x, we see by *inverting* the map T^1 that one unit of time ago, we had to be in position $-x$. So it makes sense to say

$$T^{-1}(x) = -x$$

and similarly

$$T^{-2}(x) = T^{-1-1}(x) = T^{-1}(T^{-1}(x)) = -(-x) = x$$

and so

$$T^t(x) = \begin{cases} x & \text{if } t \text{ is even} \\ -x & \text{if } t \text{ is odd} \end{cases}$$

irrespective of whether t is positive or negative.

Two examples of discrete-time systems

Example 1

In terms of "arrows", we see this dynamics:

$$... \to x \to -x \to x \to -x \to x \to -x \to x \to ...$$

where moving by t arrows corresponds to the passage of time t.

Two examples of discrete-time systems

Example 2

Let $X = \mathbb{R}$ and let $T^1(x) = x + 1$.
Then $T^t(x) = x + t$ for all x and t, and the dynamics is

$$... \to x - 1 \to x \to x + 1 \to x + 2 \to ...$$

This system is also invertible.

Orbits

One goal in the study of dynamical systems is to determine when two systems are "the same" (whatever that means).
Whatever "the same" means, Example 1 and Example 2 from the previous slides are NOT the same. How can I distinguish them?

Definition

Given an invertible, discrete-time dynamical system (X, T^t), the *orbit* of a point x is the set of all points which are of the form $T^t(x)$ for some time t (positive, negative or zero). Symbolically, we write

$$O(x) = \cup_{t \in \mathbb{Z}} T^t(x).$$

Dynamical systems that are "the same" should have the same kinds of orbits.

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Example 1 revisited

In example 1 ($X = \mathbb{R}$, $T^1(x) = -x$), all orbits are finite:

$$O(x) = \{x, -x\}$$

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Example 2 revisited

In example 2 ($X = \mathbb{R}$, $T^1(x) = x + 1$), all orbits are infinite:

$$O(x) = \{..., x-2, x-1, x, x+1, x+2, ...\}$$

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For a generic discrete-time, invertible dynamical system, orbits of some points may be finite, and orbits of other points may be infinite.

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Question

What can an orbit of a point "look like"?

Orbits in discrete-time, invertible dynamical systems

Let (X, T^t) be a discrete-time, invertible system. Then for any point x,

$$O(x) = \{..., T^{-2}(x), T^{-1}(x), x, T^1(x), T^2(x), ..\}$$

One of two things happens:

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One of two things happens:

Case 1: Two of these points coincide, say $T^t(x) = T^{s+t}(x)$

In this case, apply T^{-t} to both sides; we see that $x = T^s(x)$, i.e. x is *periodic* and therefore $O(x)$ is finite, and the dynamics on this orbit is a cyclic permutation.

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One of two things happens:

Case 2: All the points $T^t(x)$ are distinct.

In this case, the orbit is infinite:

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Better still, this orbit is *ordered* (in the same way the integers are ordered) and has an *additive structure* (in the same way that the integers do).

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So in this sense, the orbit *is a copy of* $\mathbb{Z}$.

Orbits in discrete-time, invertible dynamical systems

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Note

I think periodic orbits are dull (for example, their long-term behavior is trivial), so for the remainder of this talk I will mostly neglect periodic orbits and assume that points under consideration are non-periodic. Given this restriction, it makes sense to say that "orbits (of discrete-time, invertible systems) are copies of $\mathbb{Z}$".

Non-invertible discrete-time systems

So far, we've assumed each T^t is invertible, that is, that given a present position x, there is one (and only one) state $T^{-t}(x)$ that gives the position t units of time ago.

Non-invertible discrete-time systems

So far, we've assumed each T^t is invertible, that is, that given a present position x, there is one (and only one) state $T^{-t}(x)$ that gives the position t units of time ago.

If we don't assume the T^t are invertible, then for each $t > 0$ we define
$$T^{-t}(x) = \{y \in X : T^t(y) = x\};$$
this is a set, not a point (and is the set of all possible places you could have been t units of time ago if you are in x now).

What are the possible (non-periodic) orbits in this context? To answer this question, we need to rethink what an orbit is.

A better definition of orbit

Definition

Given any dynamical system (X, T^t) and any point $x \in X$, the *orbit* of x is

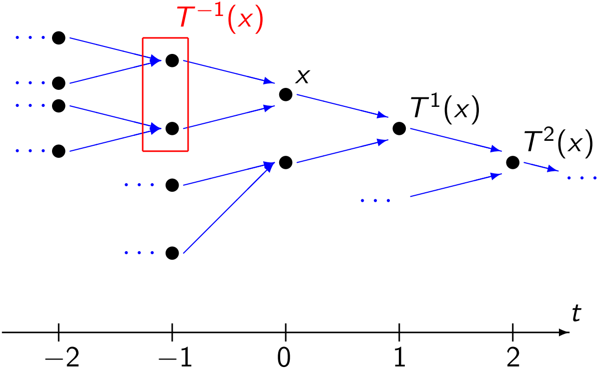
$$O(x) = \cup_{s,t \geq 0} T^{-t}(T^s(x)).$$

This is the set of points you can get to from x by first "going forward" s units of time, then "going backward" t units of time *along any possible backwards trajectory* coming from $T^s(x)$.

Given this definition, a (non-periodic) orbit in a discrete-time dynamical system is a *tree*.

Orbits in non-invertible, discrete-time systems

An example of an orbit which is a tree:



The nature of time

Recall that dynamical systems split into two types depending on whether the set of times t under consideration is *discrete* or *continuous*.

So far, we have discussed orbits in discrete-time systems.

The nature of time

Recall that dynamical systems split into two types depending on whether the set of times t under consideration is *discrete* or *continuous*.

Continuous-time dynamical systems

Here we only allow any real value of t.

$$t = 0 \leftrightarrow \text{the present}$$

$$t = 4.378 \leftrightarrow 4.378 \text{ units of time from now}$$

$$t = -\pi \leftrightarrow \pi \text{ units of time ago}$$

etc.

The nature of time

Recall that dynamical systems split into two types depending on whether the set of times t under consideration is *discrete* or *continuous*.

Continuous-time dynamical systems

As in the discrete-time case, we specify maps T^t for $t \geq 0$. If all these maps are invertible, we call the dynamical system a *flow*. In this case, $T^{-t}(x)$ is a point for every $x \in X$.

If some of the T^t are not invertible, we call the system a *semiflow*; as with non-invertible discrete systems $T^{-t}(x)$ is a set in this situation, not necessarily a point.

Example 1: translation flow

Let $X = \mathbb{R}$ and set $T^s(x) = x + s$ (for all $s \in \mathbb{R}$). This flow describes motion on $\mathbb{R}$ with constant velocity 1.

In this example, the orbit of every point is $\mathbb{R}$ (with respect to the additive structure and ordering, not necessarily the topology on $\mathbb{R}$, etc.).

Example 2: a flow coming from an ODE

Consider the differential equation

$$\begin{cases} x'(t) = -x \\ y'(t) = y \end{cases}$$

which defines a flow in the following sense: $X = \mathbb{R}^2$ and as time passes, you move along a smooth curve (parameterized by $(x(t), y(t))$) in such a way that whenever you are at the point (x, y), your velocity at that instant is $< x'(t), y'(t) > = < -x, y >$.

Example 2: a flow coming from an ODE

Again, the orbit of every point (other than the fixed point at the origin) is a copy of $\mathbb{R}$ in the sense that it is ordered and has the additive structure of $\mathbb{R}$.

Orbits in flows

Essentially, flows are like discrete-time, invertible actions: for a flow (X, T^t), orbits are either

- singletons (orbits of fixed points), or
- loops (orbits of periodic points), or
- copies of $\mathbb{R}$.

Summary so far

So far, we've seen the following "moral statements" about orbits:

- Non-periodic orbits in discrete-time, invertible dynamical systems are copies of $\mathbb{Z}$.
- Non-periodic orbits in discrete-time (not necessarily invertible) dynamical systems are trees.
- Non-periodic orbits in flows are copies of $\mathbb{R}$.

Summary so far

So far, we've seen the following "moral statements" about orbits:

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- Non-periodic orbits in flows are copies of $\mathbb{R}$.

What about orbits in semiflows? Is there an easy description of these objects?

Orbits in semiflows

Let (X, T^t) be a semiflow. Take a (non-periodic) point $x \in X$ and recall that

$$O(x) = \cup_{s,t \geq 0} T^{-t}(T^s(x)).$$

What might happen?

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$$O(x) = \cup_{s,t \geq 0} T^{-t}(T^s(x)).$$

What might happen?

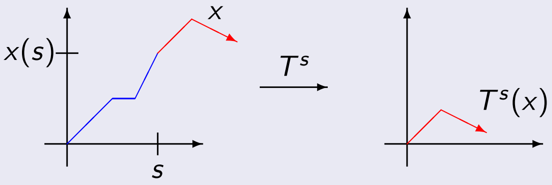
Case 1

The semiflow might be a flow. In this case, $O(x)$ is a copy of $\mathbb{R}$.

Case 2

Let $X = C_0(\mathbb{R}^+, \mathbb{R})$, the set of continuous functions from $\mathbb{R}^+ = [0, \infty)$ to $\mathbb{R}$ which pass through $(0, 0)$.

For $x = x(t) \in X$, define $(T^s(x))(t) = x(t+s) - x(s)$. This defines a semiflow on X (which models Brownian motion in one dimension).



Orbits in semiflows

Case 2

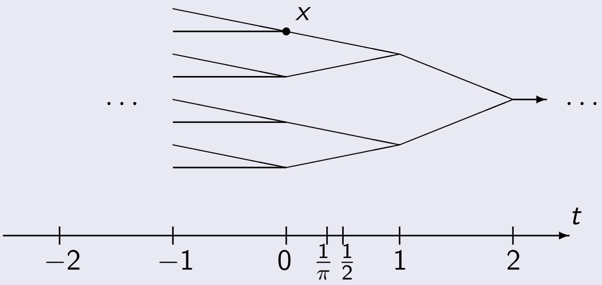
Here, $O(x)$ has a "continuous tree-like" structure.

Orbits in semiflows

Case 3

Suppose $\#(T^{-t}(x)) = 2^{n+1}$ whenever $t \in (n, n+1]$ (this holds for some suspension semiflows, for example).

Then the orbit of x has a "discrete tree-like" structure (similar to discrete-time non-invertible systems).

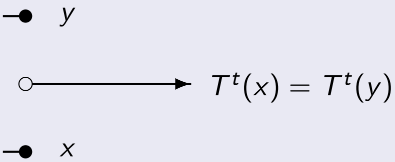


Orbits in semiflows

Case 4

In cases 1,2,3, the orbit of x is a connected object. This is not always the case for arbitrary semiflows, however.

Given x, suppose there is some point $y \neq x$ such that $T^t(y) = T^t(x)$ for all $t > 0$. Then $O(x)$ has a piece which looks like



and has a "fundamentally disconnected" structure.

Conjecture

An orbit in an arbitrary semiflow is made up of pieces which come from the cases outlined here, i.e. looks something like what I will draw on the board.

The punchline

- Non-periodic orbits in discrete-time, invertible dynamical systems are copies of $\mathbb{Z}$.
- Non-periodic orbits in discrete-time (possibly invertible) dynamical systems are trees.
- Non-periodic orbits in flows are copies of $\mathbb{R}$.
- But, orbits in semiflows are not so easily described.

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So what?

Typical problems in dynamical systems

Here are some typical problems in dynamical systems:

- Determine when two dynamical systems are "the same" (or different).
- Given a class of dynamical systems, describe a "universal model" for that class. To study the class, it is then sufficient to study the universal model.
- Realization problems, i.e. given a dynamical system in one class of systems, can it be realized in another class? (Ex: given a measurable, measure-preserving system, is it isomorphic to a continuous system? How about a smooth system?)
- Characterizing "generic" properties of dynamical systems.

Typical problems in dynamical systems

Many of these problems have been addressed for discrete-time systems and flows; the techniques to address these problems rely on the fact that the orbits of such systems are easy to characterize.

For semiflows, the analogous questions are largely unsolved.

Semiflows in the real world

Semiflows arise from situations including

- PDEs (heat equation)
- stochastic differential equations (models for commodity pricing)
- other economic models (wealth transfer across generations)
- neurology and cognitive science (how we learn things)