Ambrose-Kakutani representation theorems for Borel semiflows

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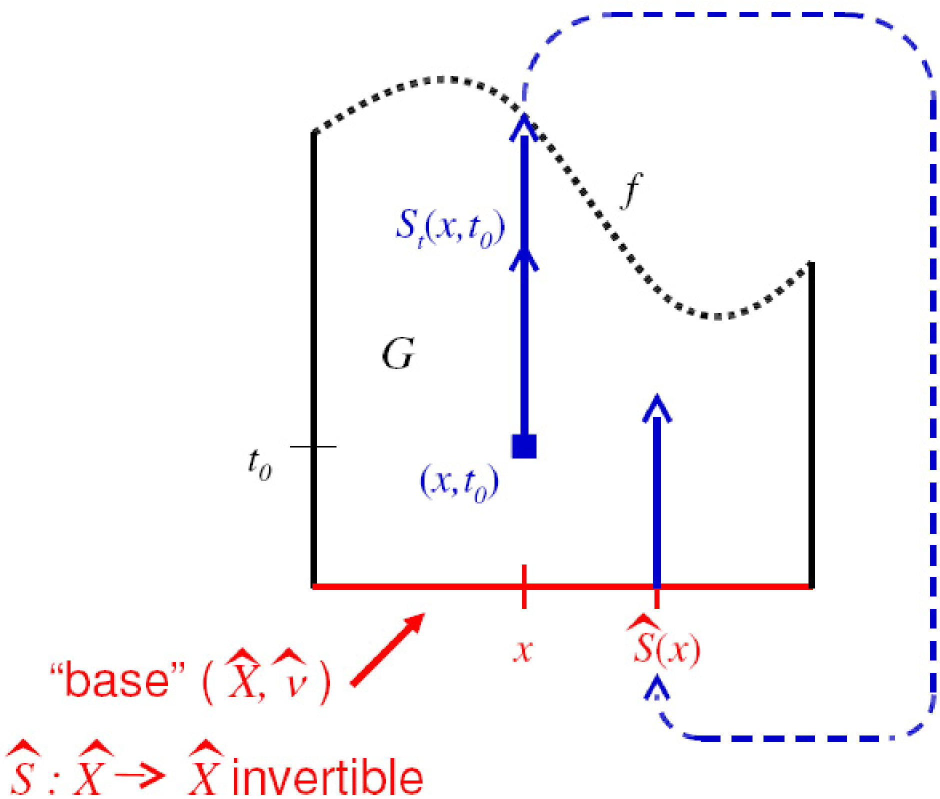
Ambrose-Kakutani Theorem

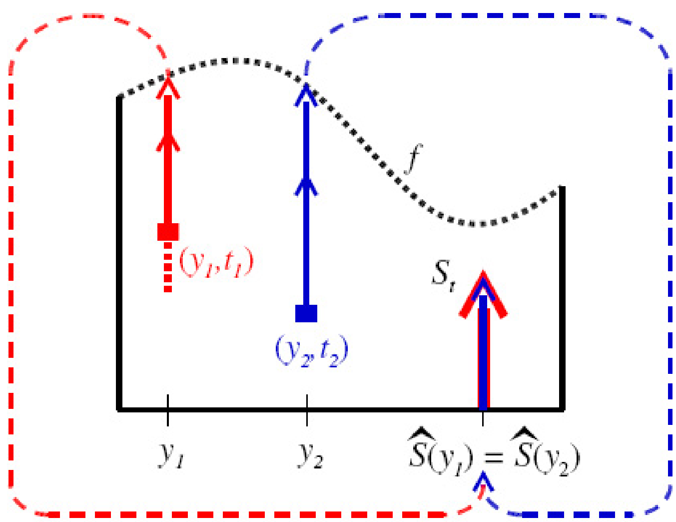
Theorem (Amb 1940, Amb-Kak 1942) *Any aperiodic measure-preserving flow T_t on a standard probability space $(X, \mathcal{X}, \mu)$ is isomorphic to a suspension flow.*

A *suspension flow* $(G, \mathcal{G}, \nu, S_t)$, also called a *flow under a function*, looks like this:



Suspension semiflows

If the return-time transformation $\widehat{S}$ in the previous picture is not injective, then we obtain a *suspension semiflow*:



Our problem

Let

- X be an uncountable Polish space, with
- $\mathcal{B}(X)$ its $\sigma-$algebra of Borel sets,
- μ a probability measure on $(X, \mathcal{B}(X))$ and
- $T_t : X \times \mathbb{R}^+ \to X$ an aperiodic, jointly Borel action by surjective maps preserving μ.

Call $(X, \mathcal{B}(X), \mu, T_t)$ a *Borel semiflow*.

Question: What Borel semiflows are isomorphic to suspension semiflows?

A restriction: discrete orbit branchings

For any point z not in the base of a suspension semiflow (G, S_t), $\#(S_{-t}(z)) = 1$ for t small enough. So if we let

$$B = \{z \in G : \#(S_{-t}(z)) > 1 \, \forall \, t > 0\},$$

every point $z \in G$ must satisfy:

The set of times $t \geq 0$ where $S_t(z) \in B$ is a discrete subset of $\mathbb{R}^+$.

More generally, we have the following for any suspension semiflow (G, S_t):

Given any z, the set of times $t_0 \geq 0$ where

$$\bigcup_{t<t_0} S_{-t}S_t(z) \neq \bigcap_{t>t_0} S_{-t}S_t(z)$$

is a discrete subset of $\mathbb{R}^+$.

Any Borel semiflow for which the preceding sentence holds is said to have *discrete orbit branchings*.

Another issue: instantaneous discontinuous identifications

Suppose (X, T_t) is a Borel semiflow and that x and y are two distinct points in X ($x \neq y$) with

$$T_t(x) = T_t(y) \,\forall\, t > 0.$$

We say that x and y are *instantaneously and discontinuously identified (IDI)* by T_t.

Define the (Borel) equivalence relation:

$$\mathbf{IDI} = \{(x,y) \in X^2 : T_t(x) = T_t(y) \,\forall t > 0\}.$$

This relation must contain the diagonal Δ. If $\mathbf{IDI} = \Delta$, we say that T_t *has no IDIs*.

T_t has no IDIs if and only if $T_{(0,\infty)}(x)$ determines x uniquely for every $x \in X$.

Suspension semiflows (as defined thus far) have no IDIs.

A conjecture

We conjecture that the previously described issues are the only restrictions to isomorphism with a suspension semiflow, i.e.

Conjecture *Any Borel semiflow with the discrete orbit branching property that has no IDIs is isomorphic to a suspension semiflow.*

A partial result

Theorem 1 *(M) If a countable-to-1 Borel semi-flow* (X, T_t) *is such that*

1. T_t *has discrete orbit branchings, and*

2. T_t *has no IDIs,*

then (X, T_t) *is isomorphic to a suspension semi-flow* (G, S_t)*, with the caveat that the measure* $\widehat{\nu}$ *on the base may be* $\sigma-$*finite.*

Note: The measure $\nu = \widehat{\nu} \times \lambda$ on G is a probability measure.

Note: Asking that T_t being countable-to-1 is virtually equivalent to asking that T_t be *bimeasurable*, that is, that $T_t(A)$ is Borel for every $t \geq 0$ and every Borel $A \subseteq X$.

An example with infinite base measure

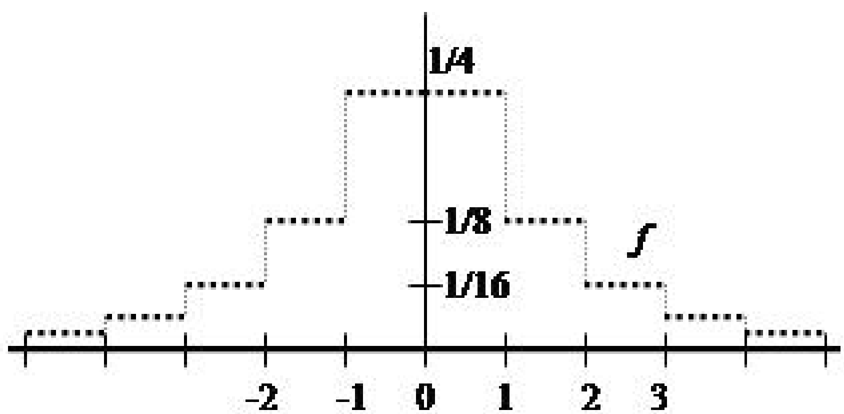
Consider the map $\widehat{S} : \mathbb{R} \to \mathbb{R}$ defined by $\widehat{S}(x) = x - \frac{1}{x}$.

Let $\widehat{X} = \mathbb{R} - \bigcup_n \widehat{S}^{-n}(0)$. (This will be the base of the suspension semiflow.)

$\widehat{S} : \widehat{X} \to \widehat{X}$ preserves Lebesgue measure, is ergodic, and is everywhere 2-to-1.

An example with infinite base measure

Construct a suspension semiflow with base $\widehat{X}$, return map $\widehat{S}$ with height function f:

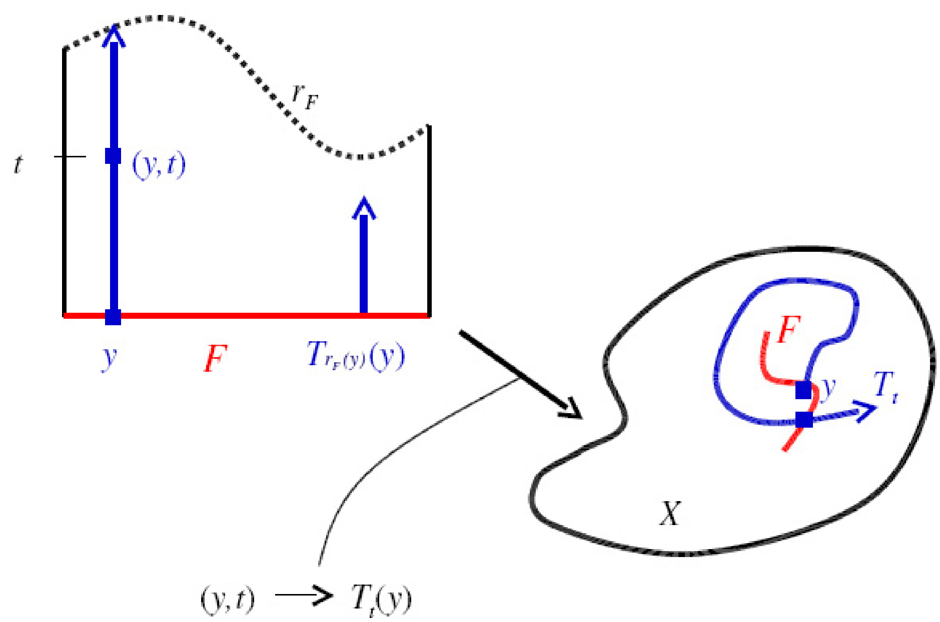


This suspension semiflow has the discrete orbit branching property but is not isomorphic to any suspension semiflow where the measure on the base is finite.

Question: What conditions ensure isomorphism with a suspension semiflow where the measure on the base is finite?

Some ingredients of the proof

Lemma 1 *(Krengel 1976, Lin & Rudolph 2002)* *Every Borel semiflow has a measurable cross-section F with measurable return-time function r_F bounded away from zero.*



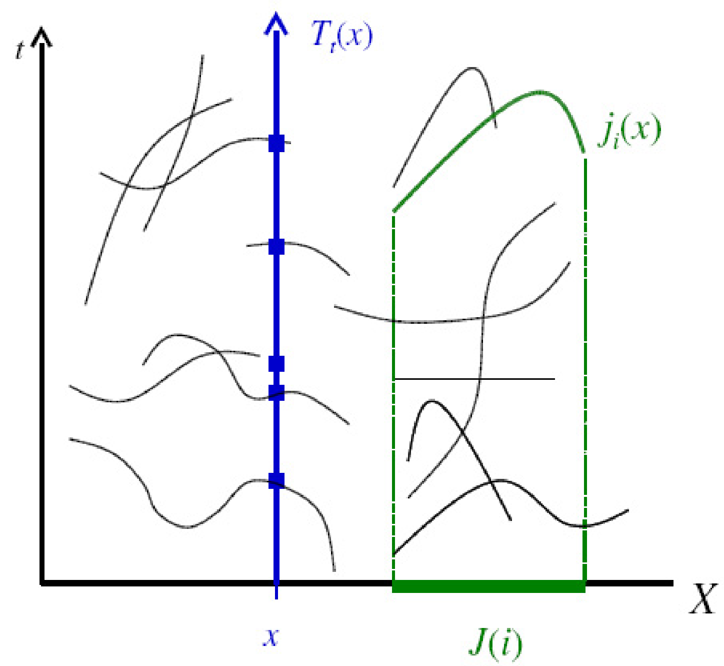
Consequence: Every $x \in X$ can be written $x = T_t(y)$ where $y \in F$ and $0 \leq t < r_F(y)$.

Another lemma

Lemma 2 *There is a countable list of Borel functions j_i taking values in $\mathbb{R}^+$ whose domains $J(i)$ are Borel subsets of X so that x has an orbit branching at time t_0, i.e.*

$$\bigcup_{t<t_0} T_{-t}T_t(z) \neq \bigcap_{t>t_0} T_{-t}T_t(z),$$

if and only if $j_i(x) = t_0$ for some i.



More on Lemma 2

To establish Lemma 2, consider the set

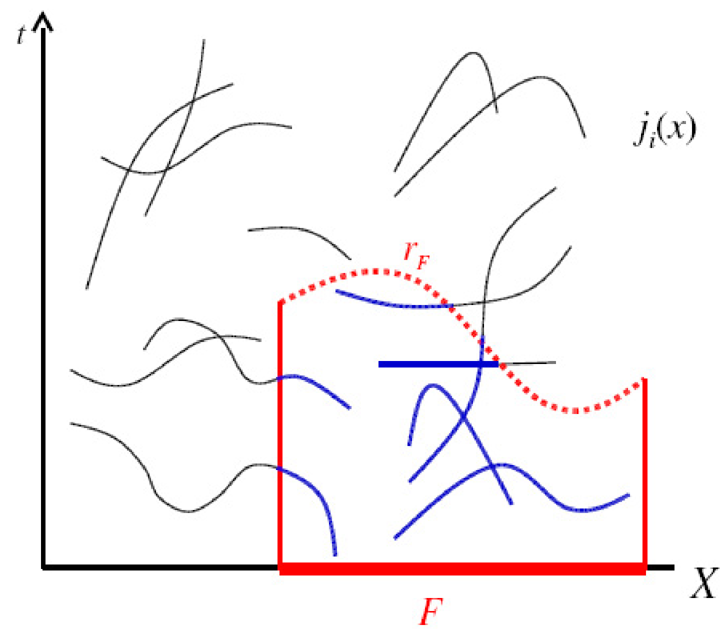
$$B^* = \{(x,t) \in X \times \mathbb{R}^+ : x \text{ has orbit branching at time } t\}.$$

Since each T_t is countable-to-1, for any Borel $A \subseteq X$, $T_t(A)$ is Borel for each $t \geq 0$. Using this, one can show that B^* is a Borel set.

Since B^* must have countable sections by assumption, the Lusin-Novikov theorem applies.

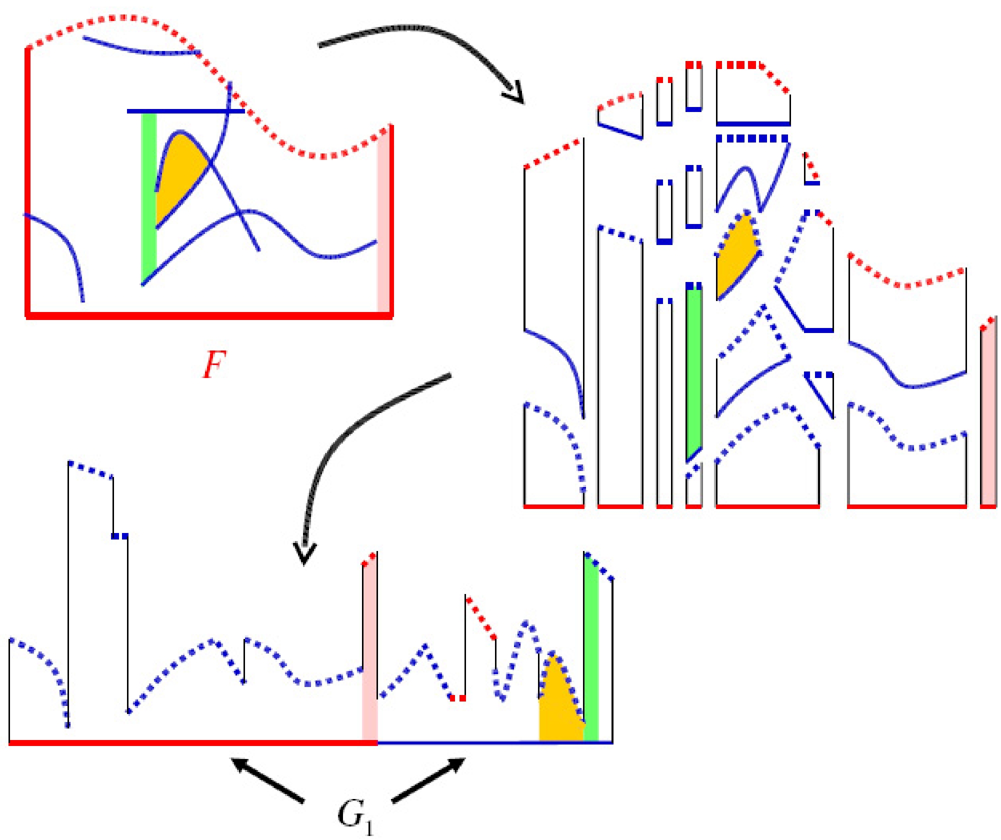
Combining the two lemmas

Superimpose the pictures from the previous two lemmas:



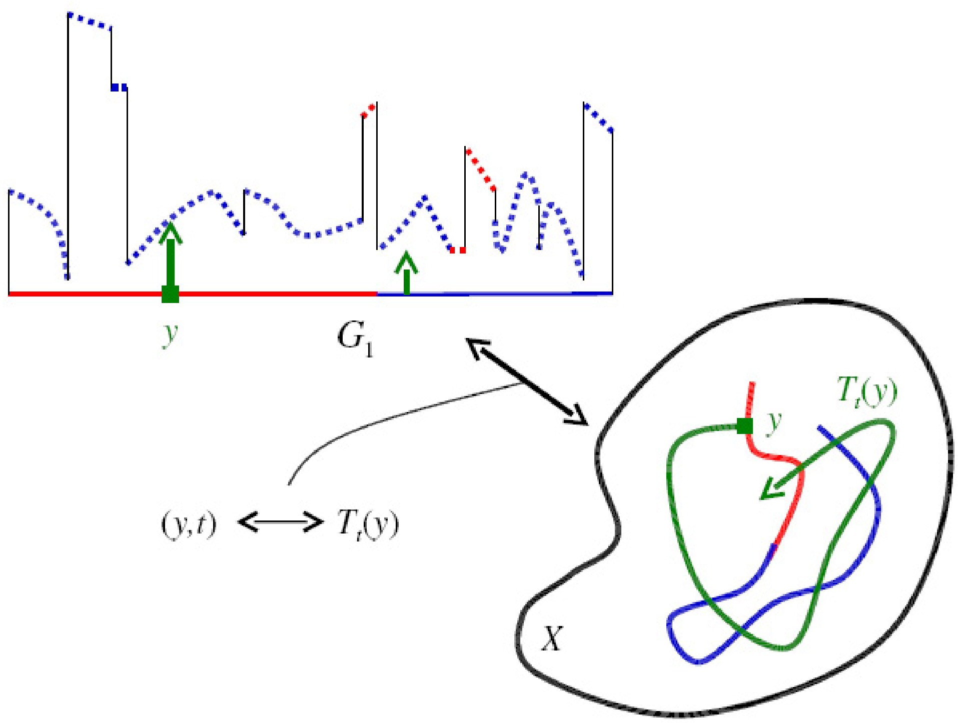
Cutting and rearranging

Make a new section G_1 (with return time function g) consisting of F together with all orbit branchings of T_t:



Obtaining an isomorphism

With respect to this new section, every $x \in X$ can be written *uniquely* as $x = T_t(y)$ where $y \in G_1$ and $0 \leq t < g(y)$. This allows for an isomorphism between (X, T_t) and the suspension semiflow over G_1.



Finite measures on the base

Theorem 2 *(M) If a countable-to-1 Borel semi-flow* (X, T_t) *is such that*

1. T_t *has no IDIs, and*

2. *there is some* $c > 0$ *such that if* x *has orbit branchings at times* t *and* t'*, then* $|t - t'| > c$*,*

then (X, T_t) *is isomorphic to a suspension semi-flow* (G, S_t) *where the measure on the base is finite.*

Proof: Adapt the preceding argument to construct a section G_1 with return-time function bounded away from zero.

What if the semiflow has IDIs?

Definition: Start with the following:

1. Two standard Polish spaces G_1 and G_2.
2. A $\sigma-$finite Borel measure $\widehat{\nu}$ on $G_1 \bigcup G_2$.
3. A measurable function $g : G_1 \to \mathbb{R}^+$ with $\int g \, d\widehat{\nu} = 1$.
4. A measurable map $\sigma : G_1 \bigcup G_2 \to G_1$ such that $\sigma|_{G_1} = id$.
5. A measurable map $\widehat{S} : G_1 \to G_1 \bigcup G_2$.

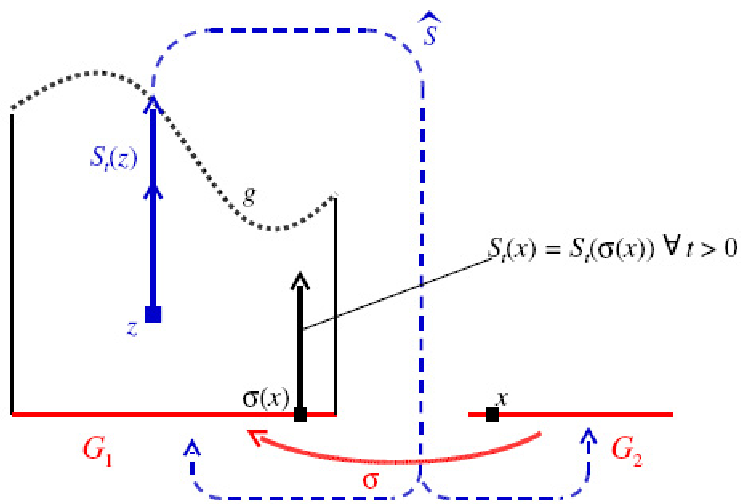
Now let G be the set

$$\Big\{(z,t) \in G_1 \times \mathbb{R}^+ : 0 \leq t < g(z)\Big\} \bigcup (G_2 \times \{0\})$$

(endowed with subspace product topology) and define the Borel semiflow S_t on G as indicated in the picture on the next slide:

Suspension semiflows with IDIs

Definition (continued):



(G, S_t) is called a *suspension semiflow with IDIs*.
Notice that for any $x \in G_2$, $(x, \sigma(x)) \in$ **IDI**.

An Ambrose-Kakutani type theorem with IDIs

Theorem 3 *(M) A countable-to-1 Borel semi-flow* (X, T_t) *is isomorphic to a suspension semi-flow with* **IDIs** *if and only if* T_t *has discrete orbit branchings.*

Questions

Suppose one considered a Borel semiflow that is not necessarily countable-to-1.

Q1. Is the discrete orbit branching property sufficient to guarantee isomorphism with a suspension semiflow with IDI?

Q2. How complicated can the **IDI** relation be? In particular, when does the relation **IDI** have a Borel selector?

- Always?
- If the semiflow has discrete orbit branchings?

More questions

Q3. Given a Borel semiflow (X, T_t), can one choose a Polish topology on X with the same Borel sets as the original topology such that the action T_t is jointly continuous?

A3. No, if $\mathbf{IDI} \neq \Delta$.

Conjecture *If T_t has no IDIs, then Q3 has an affirmative answer.*

Theorem 4 *(M) For countable-to-1 Borel semiflows with discrete orbit branchings and no IDIs, the conjecture holds.*