LEGO and Mathematics

Jonathon Wilson

Ferris State University
Big Rapids, MI, USA

joint with David McClendon

Overall Question

Overall question

How many ways can you connect n LEGO bricks of the same size and color together?

Example

How many different ways do you think there are to connect eight 4×2 standard LEGO bricks?

Overall Question

Overall question

How many ways can you connect n LEGO bricks of the same size and color together?

Example

How many different ways do you think there are to connect eight 4×2 standard LEGO bricks?

Answer

$8,274,075,616,387$ ways
(computed by Durhuus and Eilers in 2010).

Who cares?

1. Me (duh)
2. Dr. McClendon (duh)
3. You (otherwise, why are you here?)
4. Recreational mathematicians
5. Computer scientists

Why mathematicians care

Developing new techniques to count any type of structure might be useful in other contexts.

Why computer scientists care

To count the structures well, we have to divide them into types and count each type (and each type is usually counted recursively). This is kind of like writing a program that has a lot of IFs and loops in it.

Why is this difficult?

1. The number of connections gets quite large quite fast.
2. Non-Markovian.

Example: 4×2 bricks

n	# of buildings made from n 4×2 bricks
1	1
2	24
3	1,560
4	119,580
5	10,116,403
6	915,103,765
7	85,747,377,755
8	8,274,075,616,387

Some Notation

Our counting function T_B

First, we define a function $T_B(n)$ to be the number of ways we can connect n bricks of type B together.

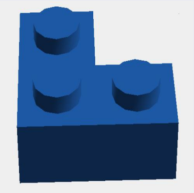
Main mathematical question

What type of function is $T_B(n)$? Linear? Exponential? Superexponential? If exponential, what is the base?

Remark

For now, we do not count the same building twice if it has just been translated and/or rotated.

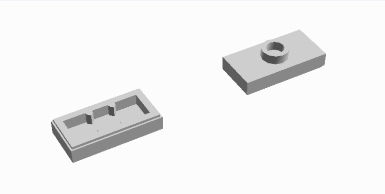
1. Durhuus-Eilers (2014) studied growth rate of $T_{b \times w}(n)$ for $b \times w$ rectangular LEGO bricks (lots of specifics in the special case 2×4; their work carries over to any standard rectangular brick)
2. McClendon-W (2017) adapted the Durhuus-Eilers work to study $T_L(n)$ for L-shaped LEGO bricks



This talk is about jumper plates

What is a jumper plate?

Here are two pictures of a jumper plate, which we call class $\mathcal{J}$:



The bottom (left) and top (right) of a LEGO jumper plate. We assume throughout that any building is rotated so that the studs of the jumper plates point up.

Parents and children

When two jumper plates are connected, we call the plate on the top the **parent** and the plate on the bottom the **child**.

Main question

Let $T_{\mathcal{J}}(n)$ be the number of buildings made from n jumper plates. What is the behavior of $T_{\mathcal{J}}(n)$?

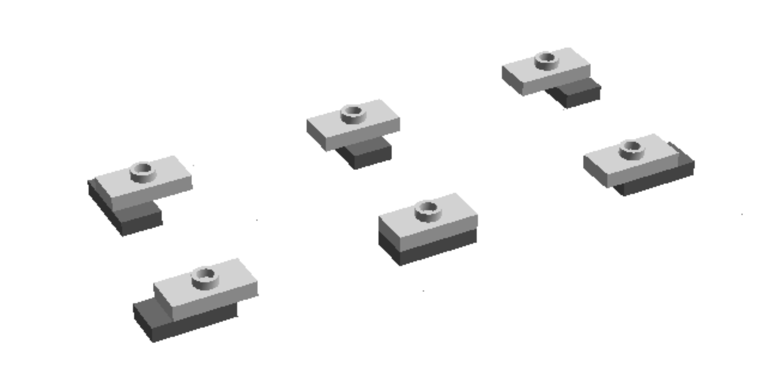
Remark

Since a jumper plate has only one stud on its top, in any building made from jumper plates there must be a unique plate in the top-most layer of the building. This plate is called the **root** of the building.

To be precise, we count the number of buildings where the root occupies a fixed position. This identifies buildings up to translation, but not rotation.

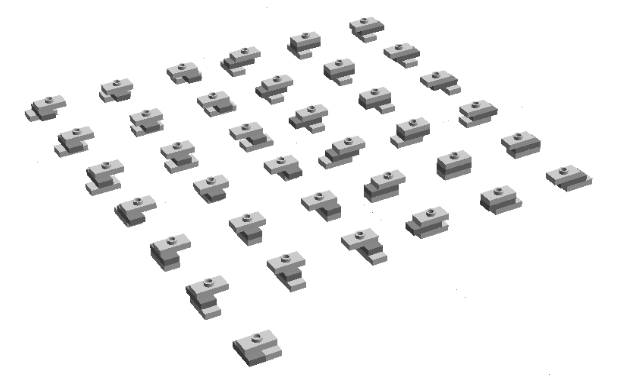
Small values of n

$T_{\mathcal{J}}(2) = 6$:



Small values of n, continued

$T_{\mathcal{J}}(3) = 37$:



Less small values of n

Values of $T_{\mathcal{J}}(n)$ for $n \leq 14$

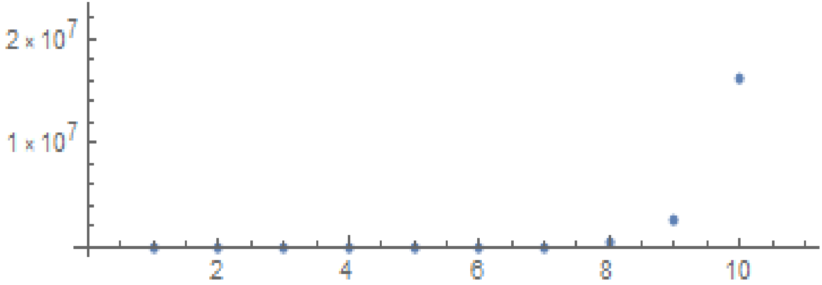
n	$T_{\mathcal{J}}(n)$
4	234
5	1489
6	9534
7	61169
8	393314 $\leftarrow$ up to here, we did these by hand
9	$2,531,777$ $\leftarrow$ from here on, Søren Eilers found these via computer and shared his counts with us
10	$16,316,262$
11	$105,237,737$
12	$679,336,650$
13	$2,194,159,545$
14	$14,183,197,852$ $\leftarrow$ after this, known computer programs take too long

A graph of $T_{\mathcal{J}}$

To get an idea of what kind of function $T_{\mathcal{J}}$ is, let's graph the points and see what we get:

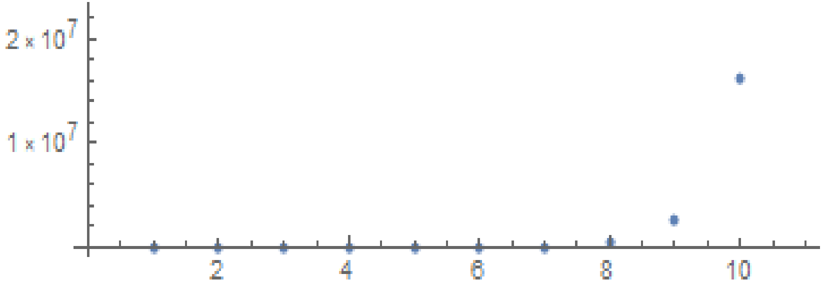
A graph of $T_{\mathcal{J}}$

To get an idea of what kind of function $T_{\mathcal{J}}$ is, let's graph the points and see what we get:



A graph of $T_{\mathcal{J}}$

To get an idea of what kind of function $T_{\mathcal{J}}$ is, let's graph the points and see what we get:

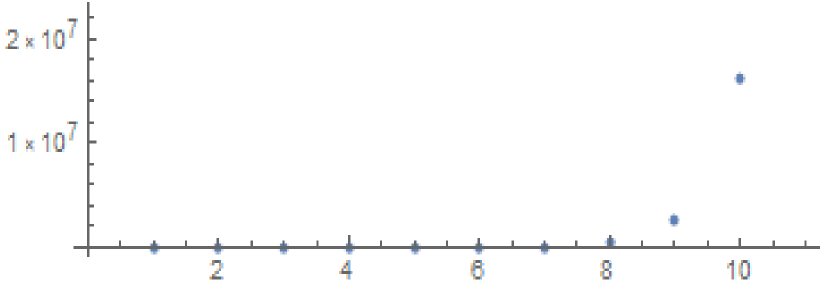


Question

What kind of a function does this look like? Linear? Polynomial? Exponential? Superexponential?

A graph of $T_{\mathcal{J}}$

To get an idea of what kind of function $T_{\mathcal{J}}$ is, let's graph the points and see what we get:

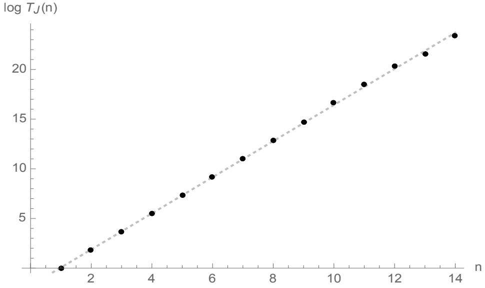


Conjecture

It looks exponential (or perhaps superexponential).

A graph on a log scale

To distinguish between exponential and superexponential behavior, we graph $\log T_{\mathcal{J}}(n)$ against n (by the way, log means base e):



Since this is appears to be roughly linear, this suggests that

$$\log T_{\mathcal{J}}(n) \text{ is linear } \Rightarrow T_{\mathcal{J}}(n) \text{ is exponential.}$$

More notation

Recall that $T_{\mathcal{J}}(n)$ is the number of ways to connect n jumper plates together.

Definition of entropy

Define the **entropy** of a jumper plate as follows:

$$h_{\mathcal{J}} := \lim_{n\to\infty} \frac{1}{n} T_{\mathcal{J}}(n)$$

What does entropy mean?

If the entropy of a brick is h, then for n large, $T_{\mathcal{J}}(n) \approx Ce^{hn}$, so the entropy h gives the exponential growth rate of $T_{\mathcal{J}}$.

Existence of the entropy

We defined:

$$h_{\mathcal{J}} := \lim_{n \to \infty} \frac{1}{n} T_{\mathcal{J}}(n)$$

Problem

Just because you write down a limit does not mean that limit exists (Math 220).

Existence of the entropy

We defined:

$$h_{\mathcal{J}} := \lim_{n\to\infty} \frac{1}{n} T_{\mathcal{J}}(n)$$

Solution

Rigorously **prove** that the limit must exist!

Existence of the entropy

We defined:

$$h_{\mathcal{J}} := \lim_{n \to \infty} \frac{1}{n} T_{\mathcal{J}}(n)$$

How to prove this limit exists

1. Write down another sequence $\{a_n\}$.
2. Use something called "Fekete's lemma" to show that $\lim_{n\to\infty} \log \frac{a_n}{n}$ exists.
3. Show that the limit in Step 2 is the entropy $h_{\mathcal{J}}$.

Existence of the entropy

We defined:

$$h_{\mathcal{J}} := \lim_{n\to\infty} \frac{1}{n} T_{\mathcal{J}}(n)$$

Lemma (Fekete 1923)

If $\{x_n\}$ is a **superadditive** sequence, i.e. the sequence satisfies $x_{m+n} \geq x_m + x_n$ for all m and n, then

$$\lim_{n\to\infty} \frac{x_n}{n}$$

exists.

Dr. McClendon says that if/when I take Math 430, I'll be able to understand the proof of this lemma.

Existence of the entropy

We defined:

$$h_{\mathcal{J}} := \lim_{n \to \infty} \frac{1}{n} T_{\mathcal{J}}(n)$$

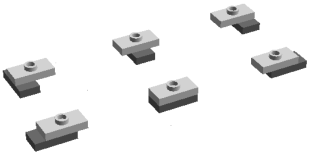
Technicality

When we say this limit "exists", we are including the possibility that the limit has value ∞.
What we are really ruling out is the possibility that this limit DNE due to oscillation (like $\lim_{x \to \infty} \sin x$).

Lower bound on $h_{\mathcal{J}}$

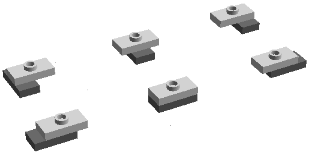
At this point we know $h_{\mathcal{J}}$ exists (in $[0, \infty]$). Now we turn to estimating its value. First, a lower bound:

Recall that there were 6 ways to connect 2 bricks together.



Lower bound on $h_{\mathcal{J}}$

Recall that there were 6 ways to connect 2 bricks together.



Therefore there are 6^{n-1} buildings **of height** n made from n jumper plates, so

$$T_{\mathcal{J}}(n) \geq 6^{n-1}$$

and therefore

$$h_{\mathcal{J}} \geq \lim_{n\to\infty} \frac{1}{n} 6^{n-1} = \log 6.$$

Lower bound on $h_{\mathcal{J}}$

But we can do better than this trivial lower bound:

Theorem (McClendon-W)

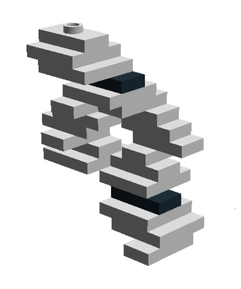
$$h_{\mathcal{J}} \geq \log 6.44947$$

How we prove that $h_{\mathcal{J}} \geq \log 6.44947$

Definition

A **bottlenecked construction** is a building that has a layer (other then the top or bottom) with only one brick in it.

Example with two bottlenecks

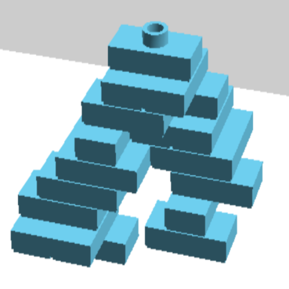


How we prove that $h_{\mathcal{J}} \geq \log 6.44947$

Definition

A **bottlenecked construction** is a building that has a layer (other then the top or bottom) with only one brick in it.

Example with no bottlenecks



How we prove that $h_{\mathcal{J}} \geq \log 6.44947$

Definition

A **bottlenecked construction** is a building that has a layer (other then the top or bottom) with only one brick in it.

Now, for each n, let c_n be the number of contiguous buildings made from $n+1$ jumper plates such that:

- the building has no bottlenecks; and
- the building has only one jumper plate on its bottom level.

Definition

A **bottlenecked construction** is a building that has a layer (other then the top or bottom) with only one brick in it.

Using something called a "generating function" (which is a power series where the coefficient on x^n is c_n), we can show

$$\sum_{n=1}^{\infty} c_n (e^{h_{\mathcal{J}}})^{-n} \leq 1.$$

We can count $c_1, c_2, ..., c_8$ directly (see the next slide); substituting these numbers into the above inequality gives our lower bound.

How we prove that $h_{\mathcal{J}} \geq \log 6.44947$

Small values of c_n

n	$c_n = \#$ buildings with no bottlenecks	lower bound on $h_{\mathcal{J}}$ using c-values up to this c_n
1	6	$\log 6$
2	0	$\log 6$
3	12	$\log 6.30214$
4	0	$\log 6.30214$
5	156	$\log 6.38779$
6	0	$\log 6.38779$
7	2652	$\log 6.42072$
8	144 $\leftarrow$ up to here, c_n computed by hand	$\log 6.42009$
9	59100 $\leftarrow$ need computer	$\log 6.43793$
10	18192	$\log 6.43872$
11	1615740	$\log 6.44947$
12	computer takes too long	

Upper bound on $h_{\mathcal{J}}$

Theorem (McClendon-W

$$h_{\mathcal{J}} \leq \log(6 + \sqrt{2})$$

To prove the lower bound (previous slides), we borrowed heavily from previous work of Durhuus and Eilers.

To prove this upper bound, we came up with entirely new stuff. The best upper bound obtainable from previously known methods is $\log 8$.

How we prove that $h_{\mathcal{J}} \leq \log(6+\sqrt{2})$

The first thing we need to talk about is trees

How we prove that $h_{\mathcal{J}} \leq \log(6+\sqrt{2})$

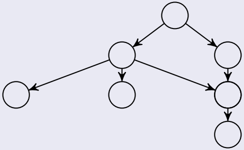
Math trees

How we prove that $h_{\mathcal{J}} \leq \log(6+\sqrt{2})$

Definition

A **graph** is a collection of points called **nodes**; some of the nodes are connected to one another by **edges**. We consider **directed graphs**, which means that the edges are like arrows as opposed to line segments. We only allow at most one arrow from one node to another, and we require that our graphs are connected.

Example (of a graph)

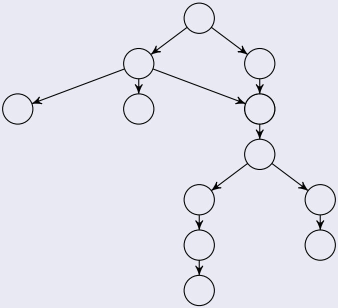


How we prove that $h_{\mathcal{J}} \leq \log(6+\sqrt{2})$

Definition

In math, a **tree** is a graph that has no loops (when defining a "loop", ignore the direction of the arrows).

Example (of a graph that isn't a tree)

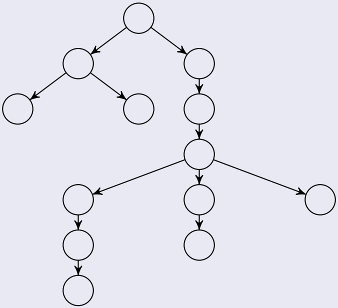


How we prove that $h_{\mathcal{J}} \leq \log(6+\sqrt{2})$

Definition

In math, a **tree** is a graph that has no loops (when defining a "loop", ignore the direction of the arrows).

Example (of a tree)



How we prove that $h_{\mathcal{J}} \leq \log(6+\sqrt{2})$

Definition

In math, a **tree** is a graph that has no loops (when defining a "loop", ignore the direction of the arrows).

Definition

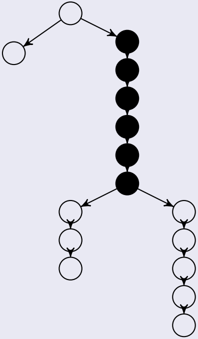
A **binary tree** is a tree such that every node in the tree has at most two children. (One node is a **child** of another if there is an edge pointing from the parent to the child.)

How we prove that $h_{\mathcal{J}} \leq \log(6+\sqrt{2})$

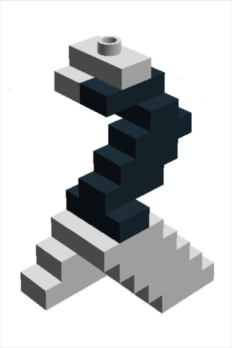
Our awesome idea

Binary trees can be used as directions to build buildings made from jumper plates:

Example



$\longrightarrow$



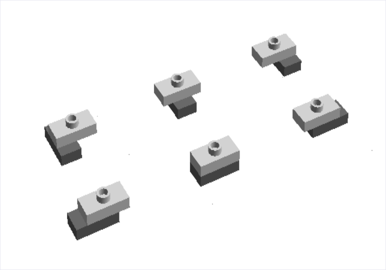
There are two potential problems with this:

Problem # 1

A tree can go with more than one building.

Example

$\longrightarrow$



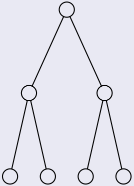
How we prove that $h_{\mathcal{J}} \leq \log(6+\sqrt{2})$

There are two potential problems with this:

Problem # 2

Some binary trees lead to no buildings. Call a binary tree **allowable** if at least one building can be made from it (using jumper plates) in the physical world.

Example of a nonallowable tree



$\longrightarrow$ nothing you can build with jumper plates (visualize or try it)

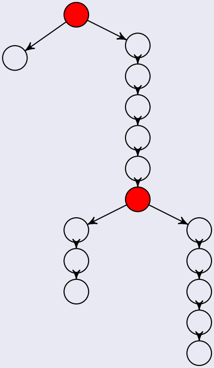
How we prove that $h_{\mathcal{J}} \leq \log(6+\sqrt{2})$

There are two potential problems with this:
To fix these problems, we ...

1. ... find an upper bound on the number of buildings that can be made from each allowable tree (accounts for Problem # 1), and ...
2. ... find an upper bound on the number of allowable binary trees (accounting for Problem # 2).

Fixing Problem # 1

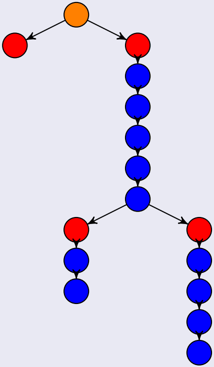
To count an upper bound on the number of buildings that can be made from each allowable tree, count the number of **branchings** in the tree:



This tree has 16 nodes
and 2 branchings
(at the red nodes).

Fixing Problem # 1

To count an upper bound on the number of buildings that can be made from each allowable tree, count the number of **branchings** in the tree:



This tree has 16 nodes
and 2 branchings
(at the red nodes).

So it can be turned into
(at most)
$6^{16-1-2(2)} = 6^{11}$ buildings.

How we prove that $h_{\mathcal{J}} \leq \log(6+\sqrt{2})$

Definition

Let $Q(n,k)$ as the number of allowable binary trees with exactly n nodes and exactly k branchings.

Each tree with n nodes and k branchings can be turned into at most 6^{n-1-2k} buildings, so:

What we know at this point

$$T_{\mathcal{J}}(n) \leq \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} 6^{n-1-2k} Q(n,k).$$

($\lfloor \frac{n-1}{2} \rfloor$ is the maximum number of branchings a binary tree with n nodes can have.)

What we know at this point

$$T_{\mathcal{J}}(n) \leq \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} 6^{n-1-2k} Q(n,k).$$

What we need to do now

Find an upper bound on $Q(n,k)$ (this will fix Problem # 2).

How we prove that $h_{\mathcal{J}} \leq \log(6+\sqrt{2})$

Remember that our goal is to find an upper bound on $Q(n,k)$, the number of allowable binary trees with n nodes and k branchings. To do this, we prove a lot of crap about $Q(n,k)$:

Lemma (Properties of $Q(n,k)$)

Let $Q(n,k)$ be defined as above. Then:

1. *If $n < 2k+1$, then $Q(n,k) = 0$.*
2. *For any $n \in \{1,2,3,\ldots\}$, $Q(n,0) = 1$.*
3. *For any $k \in \{1,2,\ldots\}$, $Q(2k+1,k) = 2^{k-1}$.*

How we prove that $h_{\mathcal{J}} \leq \log(6+\sqrt{2})$

Remember that our goal is to find an upper bound on $Q(n,k)$, the number of allowable binary trees with n nodes and k branchings. To do this, we prove a lot of crap about $Q(n,k)$:

Lemma (Recursive upper bound for $Q(n,k)$)

For any $n \in \{1,2,3,\ldots\}$ and any $k \in \{0,1,2,\ldots\}$,

$$Q(n,k) \leq Q(n-1,k) + \sum_{j=0}^{n-1}\sum_{s=0}^{k-1} Q(j,s)Q(n-j-1,k-s-1).$$

Aside: combinations

If we have n objects and wish to choose k of them (where the order in which they're picked doesn't matter), the number of ways to do this is

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}.$$

We pronounce this number as "n **choose** k". In Math 414, you learn lots of stuff about these numbers.

How we prove that $h_{\mathcal{J}} \leq \log(6+\sqrt{2})$

Putting our lemmas together, we can prove this:

Lemma (Upper bound on $Q(n,k)$)

Let $Q(n,k)$ be defined as above. Then

$$Q(n,k) \leq \binom{n-1}{2k} 2^{k-1}.$$

The proof is by induction (the base case uses the first lemma I wrote down; the induction step uses the recursive upper bound).

How we prove that $h_{\mathcal{J}} \leq \log(6+\sqrt{2})$

We're almost there, I promise!

Next up:

Lemma

For any $r \in (0,1)$,

$$\sum_{k=0}^{\infty} \binom{n-1}{2k} r^k = \frac{(1+\sqrt{r})^{n-1} + (1-\sqrt{r})^{n-1}}{2}.$$

To prove this, expand the right-hand side with the Binomial Theorem, which says

$$(1+r)^n = \sum_{k=0}^{\infty} \binom{n}{k} r^k,$$

and manipulate the resulting stuff to get the left-hand side.

How we prove that $h_{\mathcal{J}} \leq \log(6 + \sqrt{2})$

Wrapping up the proof that $h_{\mathcal{J}} \leq \log(6 + \sqrt{2})$

X

Review

Main question

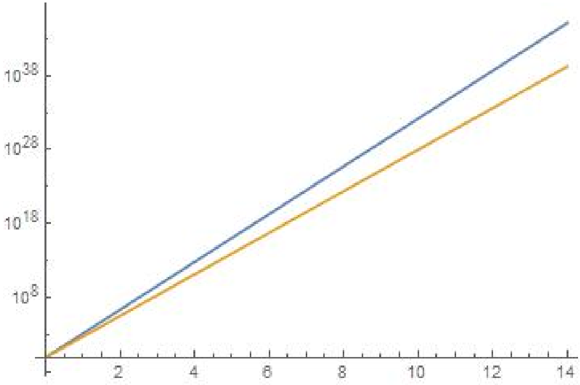
How many ways can you connect n LEGO jumper plates of the same size and color together?

Answer

Let $T_{\mathcal{J}}(n)$ be the number of ways to connect n bricks together. From our work, we know $T_{\mathcal{J}}$ has exponential growth rate, and this rate is between

$$e^{6.44947} \text{ and } e^{(6+\sqrt{2})} \approx e^{7.41421}.$$

This gives us a window of something like this:



Now for the hard stuff

No, I'm not joking.
(roof tiles)

What did we need to get these results?

1. Combinatorics: binomial theorem, combinations (MATH 328, 414, 251)
2. Analysis of recursive formulas (CPSC 300)
3. Calculus: infinite series, generating functions (MATH 230)
4. Real Analysis: Fekete's lemma (MATH 430)
5. Graph theory: binary trees (CPSC 300)
6. Induction proofs (MATH 324, 328)
7. Complex numbers (not at FSU $\ddot{\frown}$)
8. Time (priceless)
9. The internet (to look up others' research)
10. A little help from *Mathematica* (MATH 220, 230, 322)

Anyone in?